Analysis of RHC for stabilization of nonautonomous parabolic equations under uncertainty

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Abstract

Stabilization of a class of time-varying parabolic equations with uncertain input data using Receding Horizon Control (RHC) is investigated. The diffusion coefficient and the initial function are prescribed as random fields. We consider both cases, uniform and log-normal distributions of the diffusion coefficient. The controls are chosen to be finite dimensional and enter into the system as a linear combination of finitely many indicator functions (actuators) supported in open subsets of the spatial domain. Under suitable regularity assumptions, we study the expected (averaged) stabilizability of the RHC-controlled system with respect to the number of actuators. An upper bound is also obtained for the failure probability of RHC in relation to the choice of the number of actuators and parameters in the equation.

Keywords— receding horizon control, random evolution PDEs, uncertainty, averaged stabilizability, random fields, nonautonomous parabolic equations

Mathematics Subject Classification 93C20 · 35R60 · 93D20 · 93E03

1 Introduction

Mathematical models arising in real-world applications are typically affected by uncertainties that can lead to significant differences between the real systems response and the corresponding deterministic mathematical models. Therefore it is of great interest for applications to include uncertainty in these models and quantify its effect on the predicted quantities of interest. Such uncertainty may reflect our ignorance or inability to properly characterize all input parameters of the mathematical model, and it may also describe an intrinsic variability of the physical system, see e.g., [5, 6]. Probability theory provides a natural framework to describe and deal with such uncertainties which are characterized as random variables or more generally random fields.

We investigate stabilization of the controlled systems governed by the following linear parabolic equation utilizing the receding horizon control (RHC) strategy

$$\begin{cases} \partial_t y - \nabla \cdot (\nu(\omega)\nabla y) + a(t)y + \nabla \cdot (b(t)y) = \sum_{i=1}^{N} u_i(t)\mathbf{1}_{O_i} & (t,x) \in (0,\infty) \times D, \\ y = 0 & (t,x) \in (0,\infty) \times \partial D, \\ y(0) = y_0(\omega) & x \in D, \end{cases} \tag{CS}$$

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where $D \subset \mathbb{R}^n$ is a bounded rectangular with boundary ∂D and the functions $\mathbf{1}_{O_i}$, represent the actuators. They are modelled as the characteristic functions related to open sets $O_i \subset D$ for $i = 1, \dots, N$. The reaction term $a(t) = a(t, x)$ and the convection term $b(t) = b(t, x)$ are, respectively, real- and $\mathbb{R}^n$-valued functions of $(t, x) \in (0, \infty) \times D$. Further, the diffusion coefficient (the convective heat transfer coefficient) $\nu(\omega) = \nu(\omega, x)$ and the initial function $y_0(\omega) = y_0(x, \omega)$ are very difficult to measure in practice and, hence, they are affected by a certain amount of uncertainty with $x \in D$ and $\omega \in \Omega$. These uncertainty inputs are described as random fields defined on the complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where Ω denotes the set of outcomes, $\mathcal{F}$ is the associated σ-algebra of events, and $\mathbb{P} : \mathcal{F} \to [0, 1]$ is a probability measure.

In this work, we aim at deriving a stabilizing control that is robust with respect to the perturbations of the dynamics caused by all possible realizations of the random parameter ω. For this purpose we consider the notion of averaged stability and verify that the expected value of the distance of the state to the steady state with respect to the random parameter converges asymptotically to zero. More concretely, the control objective is to find a (spatially) finite-dimensional control $\mathbf{u} \in L^2((0, \infty); U)$ for which

$$\mathbb{E}\left[\|y(t)\|^2_{L^2(D;\mathbb{R})}\right] := \int_\Omega \|y(t, \omega)\|^2_{L^2(D;\mathbb{R})} d\mathbb{P}(\omega) \to 0 \quad \text{as} \quad t \to \infty$$

holds. Here we will consider the both cases deterministic $U = \mathbb{R}^N$ and stochastic $U = L^2_{\mathbb{P}}(\Omega; \mathbb{R}) \otimes \mathbb{R}^N$ controls. The stabilizing control $\mathbf{u}$ is computed by a receding horizon framework. In this framework, the current control action is obtained by minimizing a performance index defined on a finite time interval, ranging from the current time t_0 to some future time $t_0 + T$, with $T \in (0, \infty]$ and $t_0 \in (0, \infty)$. Here we set

$$J_T(\mathbf{u}; t_0, y_0) := \frac{1}{2} \int_{t_0}^{t_0+T} \mathbb{E}\left[\ell(t, y(t))\right] dt + \frac{\beta}{2} \int_{t_0}^{t_0+T} \|\mathbf{u}(t)\|^2_U dt, \tag{1}$$

for $\beta \geq 0$, $\ell : \mathbb{R}_{\geq 0} \times H^1_0(D; \mathbb{R}) \to \mathbb{R}_{\geq 0}$ satisfying $\ell(t, y) \geq \alpha_\ell \|y\|^2_{L^2(D;\mathbb{R})}$ with $\alpha_\ell > 0$, and $\ell(t, 0) = 0$. As an example one may consider $\ell(t, y) = \alpha_\ell \|y\|^2_{L^2(D;\mathbb{R})}$. Then, the stabilization of the control system (CS) can be formulated as the following infinite-horizon optimal control problem

$$\min\{J_\infty(\mathbf{u}; 0, y_0) \mid (y, \mathbf{u}) \text{ satisfies (CS)}, \mathbf{u} \in L^2((0, \infty); U)\}. \qquad OP_\infty(y_0)$$

As we will show, the receding horizon approach delivers a suboptimal approximation to the solution of $OP_\infty(y_0)$. This approximation is constructed by concatenation of a sequence of finite horizon optimal controls defined on overlapping intervals covering $(0, \infty)$. These finite horizon problems have the following form. For a given initial time $\bar{t}_0$, initial functions $\bar{y}_0 = \bar{y}_0(\omega, x)$, and prediction horizon T consider

$$\min_{\mathbf{u} \in L^2((\bar{t}_0, \bar{t}_0+T); U)} J_T(\mathbf{u}; \bar{t}_0, \bar{y}_0) \qquad OP_T(\bar{t}_0, \bar{y}_0)$$

$$\text{s.t} \begin{cases} \partial_t y - \nabla \cdot (\nu(\omega) \nabla y) + a(t) y + \nabla \cdot (b(t) y) = \sum_{i=1}^N u_i(t) \mathbf{1}_{O_i} & (t, x) \in (\bar{t}_0, \bar{t}_0 + T) \times D, \\ y = 0 & (t, x) \in (\bar{t}_0, \bar{t}_0 + T) \times \partial D, \\ y(\bar{t}_0) = \bar{y}_0(\omega) & x \in D. \end{cases} \tag{2}$$

In the receding horizon framework, we define sampling instances $t_k := k\delta$, for $k = 0, 1, 2, \dots$, and for a chosen sampling time $\delta > 0$. Then, at every current sampling instance t_k sampling time $\delta > 0$. Then, at every current sampling instance t_k with state $y_{rh}(t_k) \in L^2_{\mathbb{P}}(\Omega; \mathbb{R}) \otimes L^2(D; \mathbb{R})$, an open-loop optimal control problem $OP_T(t_k, y_{rh}(t_k))$ is solved over a finite prediction horizon $[t_k, t_k + T]$ for an appropriate prediction horizon $T > \delta$. Then, the associated optimal control is applied to steer the system from time t_k with the initial state $y_{rh}(t_k) \in L^2_{\mathbb{P}}(\Omega; \mathbb{R}) \otimes L^2(D; \mathbb{R})$ until time $t_{k+1} := t_k + \delta$ at which point, a new measurement of the state $y_{rh}(t_{k+1}) \in L^2_{\mathbb{P}}(\Omega; \mathbb{R}) \otimes L^2(D; \mathbb{R})$, is assumed to be available. The process is repeated starting from this new measured state: we obtain a new optimal control and a new predicted state trajectory by shifting the prediction horizon forward in time. The sampling time δ is the time period between two sample instances. Throughout, we denote the receding horizon state- and control variables by $y_{rh}(\cdot, \cdot)$ and $\mathbf{u}_{rh}(\cdot)$, respectively. Also, $(y^*_T(\cdot, \cdot; \bar{t}_0, \bar{y}_0), \mathbf{u}^*_T(\cdot; \bar{t}_0, \bar{y}_0))$ stands for the optimal state and

Algorithm 1 Robust RHC(δ, T)

Require: The sampling time δ, the prediction horizon $T \geq \delta$, and the initial state y_0
Ensure: The stability of RHC $\mathbf{u}_{rh}$.

1: Set $(\bar{t}_0, \bar{y}_0) := (0, y_0)$ and $y_{rh}(0) = y_0$;
2: Find the solution $(y_T^*(\cdot; \bar{t}_0, \bar{y}_0)), \mathbf{u}_T^*(\cdot; \bar{t}_0, \bar{y}_0))$ over the time horizon $(\bar{t}_0, \bar{t}_0 + T)$ by solving the open-loop problem $OP_T(\bar{t}_0, \bar{y}_0)$;
3: For all $\tau \in [\bar{t}_0, \bar{t}_0 + \delta)$, set $y_{rh}(\tau) = y_T^*(\tau; \bar{t}_0, \bar{y}_0)$ and $\mathbf{u}_{rh}(\tau) = \mathbf{u}_T^*(\tau; \bar{t}_0, \bar{y}_0)$;
4: Find a measurement $y_{rh}(\bar{t}_0 + \delta; \bar{t}_0, \bar{y}_0)$ of the state at time $\bar{t}_0 + \delta$;
5: Update: $(\bar{t}_0, \bar{y}_0) \leftarrow (\bar{t}_0 + \delta, y_{rh}(\bar{t}_0 + \delta; \bar{t}_0, \bar{y}_0))$;
6: Go to Step 2;

control of the optimal control problem with finite time horizon T, and initial function $\bar{y}_0$ at initial time $\bar{t}_0$. This is summarized in Algorithm 1.

Concerning the literature, there is a growing interest in partial differential equations (PDEs) that involve some uncertainty. So far, there are only a few papers investigating parabolic PDEs with random coefficients. Here we can quote e.g., [9, 13, 14, 21, 27]. Concerning control and stabilization, which are well-studied for deterministic infinite-dimensional systems, and stochastic systems with the stochastic terms appearing in an affine manner, there is little research on infinite-dimensional systems under uncertainty. Among them, we can mention [2, 7, 17] in the context of controllability results and [11, 16, 18, 19] for optimal control problems. To our knowledge, RHC has not yet been studied for control systems with uncertainty inputs. In this project, we take a step in this direction and, relying on theoretical results in [1], we investigate the performance and stability of the receding horizon framework for eq. (CS) with both uniformly bounded and log-normally distributed random diffusions ν. For each case, separately, this involves investigating the well-posedness of the state, the stabilizability of the controlled system by (spatially) finite-dimensional controls, and deriving continuity- and observability-type of inequalities. We also provide an upper bound for the failure probability for the receding horizon framework depending on the choice of diffusion parameter ν, reaction and convection terms a and b, and the number N of actuators.

The rest of the paper is organized as follows. We start the next section by introducing the notation which is used throughout the paper. In Section 3, we consider time-varying parabolic equations with uniformly bounded random diffusion. Under appropriate assumptions, we study the well-posedness of the state equation, the stabilizability of the controlled system, and the stability of the receding horizon framework. At the end of that section, we derive an upper bound for the failure probability of RHC. In the forth section, we discuss the analogous questions for the case of log-normally distributed random diffusion.

2 Notation and preliminaries

Let Banach spaces X and Y be given. We write $X \hookrightarrow Y$ if the inclusion $X \subseteq Y$ is continuous. The space of continuous linear mappings from X into Y is denoted by $\mathcal{L}(X, Y)$. We also write $\mathcal{L}(X) := \mathcal{L}(X, X)$. The continuous dual of X is denoted by $X' := \mathcal{L}(X, \mathbb{R})$. The adjoint of an operator $L \in \mathcal{L}(X, Y)$ will be denoted with $L^* \in \mathcal{L}(Y', X')$.

Let a Hilbert space H endowed with scalar product $(\cdot, \cdot)_H$ be given. Then the orthogonal complement to a given subset $B \subset H$ is denoted by $B^\perp := \{h \in H : (h, s)_H = 0 \quad \forall s \in B\}$.

For any two closed subspaces $\mathcal{F}$ and $\mathcal{G}$ of the Hilbert space H satisfying $H = \mathcal{F} \oplus \mathcal{G}$, we define by $P_{\mathcal{F}}^{\mathcal{G}} \in \mathcal{L}(H, \mathcal{F})$ the oblique projection in H onto $\mathcal{F}$ along $\mathcal{G}$. That is, for every $h \in H$ if we consider the unique decomposition $h = h_{\mathcal{F}} + h_{\mathcal{G}}$ with $h_{\mathcal{F}} \in \mathcal{F}$ and $h_{\mathcal{G}} \in \mathcal{G}$, we have $P_{\mathcal{F}}^{\mathcal{G}} h := h_{\mathcal{F}}$. Then, clearly, $P_{\mathcal{F}}^{\mathcal{F}^\perp}$ is the orthogonal projection in H onto $\mathcal{F}$ which is denoted by $P_{\mathcal{F}}$.

For given Hilbert spaces H_1 and H_2, we use the notation $H_1 \otimes H_2$ for the tensor product of H_1 with H_2.

We also consider the spaces $V := H_0^1(D;\mathbb{R})$, $V' := H^{-1}(D;\mathbb{R})$, and $H := L^2(D;\mathbb{R})$ endowed with their usual norms. Then, for every open interval $(s_1, s_2) \subset \mathbb{R}_{\geq 0}$, we can define

$$W(s_1, s_2) := \{v \in L^2((s_1, s_2); V) : \partial_t v \in L^2((s_1, s_2); V')\},$$

endowed with the norm $\|v\|_{W(s_1,s_2)} := \left(\|v\|^2_{L^2((s_1,s_2);V)} + \|\partial_t v\|^2_{L^2((s_1,s_2);V')}\right)^{\frac{1}{2}}$.

For the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, a Banach space X, and $p \in [1, \infty]$, we denote by $L^p_{\mathbb{P}}(\Omega; X)$ the Lebesgue-Bochner space, composed of all strongly measurable function $v : \Omega \to X$ whose norm is defined by

$$\|v\|_{L^p_{\mathbb{P}}(\Omega;X)} := \begin{cases} \left(\int_\Omega \|v(\cdot,\omega)\|^p_X d\mathbb{P}(\omega)\right)^{\frac{1}{p}} & p < \infty, \\ \operatorname{ess\,sup}_{\omega\in\Omega} \|v(\cdot,\omega)\|_X & p = \infty. \end{cases}$$

We also assume that $L^2_{\mathbb{P}}(\Omega;\mathbb{R})$ is a separable Hilbert space. For this assumption it suffices to assume that $(\Omega, \mathcal{F}, \mathbb{P})$ is separable see e.g., [22, Theorem II.10]. Then, if $p = 2$ and X is a separable Hilbert space, the Bochner space $L^2_{\mathbb{P}}(\Omega; X)$ is isomorphic to the tensor product space $L^2_{\mathbb{P}}(\Omega;\mathbb{R}) \otimes X$, that is $L^2_{\mathbb{P}}(\Omega; X) \cong L^2_{\mathbb{P}}(\Omega;\mathbb{R}) \otimes X$, see e.g., [3, Theorem 4.13].

For the sake of convenience, we abbreviate $V_{\mathbb{P}} := L^2_{\mathbb{P}}(\Omega; H_0^1(D;\mathbb{R}))$, $V'_{\mathbb{P}} := L^2_{\mathbb{P}}(\Omega; H^{-1}(D;\mathbb{R}))$, $H_{\mathbb{P}} := L^2_{\mathbb{P}}(\Omega; L^2(D;\mathbb{R}))$ and $U^N_{\mathbb{P}} := L^2_{\mathbb{P}}(\Omega;\mathbb{R}^N)$. By identifying $H_{\mathbb{P}}$ with its dual we obtain a Gelfand triple $V_{\mathbb{P}} \hookrightarrow H_{\mathbb{P}} \hookrightarrow V'_{\mathbb{P}}$ of separable Hilbert spaces with dense injections. Finally, for every open interval $(s_1, s_2) \subset \mathbb{R}_{\geq 0}$, we consider the space $W_{\mathbb{P}}(s,t)$ defined

$$W_{\mathbb{P}}(s_1, s_2) := \{v \in L^2((s_1, s_2); V_{\mathbb{P}}) : \partial_t v \in L^2((s_1, s_2); V'_{\mathbb{P}})\},$$

and endowed with the norm $\|v\|_{W_{\mathbb{P}}(s_1,s_2)} := \left(\|v\|^2_{L^2((s_1,s_2);V_{\mathbb{P}})} + \|\partial_t v\|^2_{L^2((s_1,s_2);V'_{\mathbb{P}})}\right)^{\frac{1}{2}}$. It is well known that $W_{\mathbb{P}}(s_1, s_2) \hookrightarrow C([s_1, s_2]; H_{\mathbb{P}})$. Further, due to the fact that $L^2_{\mathbb{P}}(\Omega;\mathbb{R})$ is separable, we can write for any Hilbert space X that

$$\begin{aligned} L^2_{\mathbb{P}}(\Omega;\mathbb{R}) \otimes L^2((s_1, s_2); X) &\cong L^2_{\mathbb{P}}(\Omega; L^2((s_1, s_2); X)) \cong L^2(\Omega \times (s_1, s_2); X) \\ &\cong L^2((s_1, s_2); L^2_{\mathbb{P}}(\Omega; X)) \cong L^2((s_1, s_2);\mathbb{R}) \otimes L^2_{\mathbb{P}}(\Omega; X). \end{aligned}$$

Hence, we can conclude

$$L^2_{\mathbb{P}}(\Omega;\mathbb{R}) \otimes W(s_1, s_2) \cong W_{\mathbb{P}}(s_1, s_2). \tag{3}$$

In the following, we define the finite- and infinite-horizon value functions. These will be used frequently in the analysis of RHC.

Definition 2.1. *For any $y_0 \in H_{\mathbb{P}}$ the infinite-horizon value function $V_\infty : H_{\mathbb{P}} \to \mathbb{R}_{\geq 0}$ is defined by*

$$V_\infty(y_0) := \min_{\mathbf{u}\in L^2((0,\infty);U)} \{J_\infty(\mathbf{u}; 0, y_0) \text{ subject to (CS)}\}.$$

Similarly, for every $(T, \bar{t}_0, \bar{y}_0) \in \mathbb{R}^2_{\geq 0} \times H_{\mathbb{P}}$, the finite-horizon value function $V_T : \mathbb{R}_{\geq 0} \times H_{\mathbb{P}} \to \mathbb{R}_{\geq 0}$ is defined by

$$V_T(\bar{t}_0, \bar{y}_0) := \min_{\mathbf{u}\in L^2((\bar{t}_0,\bar{t}_0+T);U)} \{J_T(\mathbf{u}; \bar{t}_0, \bar{y}_0) \text{ subject to (2)}\}.$$

3 Parabolic PDEs with uniform random diffusion

In this section we are concerned with the case when the diffusion coefficient is uniformly bounded away from zero and from above. This allows us to use the weak formulation directly. Throughout this section, we impose the following conditions:

Assumption 3.1. *We assume that:*

A1: There are random variables $\nu_{\min}$, $\nu_{\max}$, and constants $\overline{\nu}$, $\underline{\nu}$ such that

$$0 < \underline{\nu} \leq \nu_{\min}(\omega) \leq \nu(\omega, x) \leq \nu_{\max}(\omega) \leq \overline{\nu} < \infty \quad \text{for a.e. } x \in D \text{ and } \omega \in \Omega \text{ a.s..} \tag{4}$$

A2: For the reaction parameter a and convection vector b, we impose

$$a \in L^\infty((0,\infty); L^r(D;\mathbb{R})) \text{ with } r \geq n := dim(D), \text{ and } b \in L^\infty((0,\infty) \times D; \mathbb{R}^n), \tag{RA}$$

and set $\mathcal{N}(a,b) := \|a\|_{L^\infty((0,\infty);L^r(D;\mathbb{R}))} + \|b\|_{L^\infty((0,\infty)\times D;\mathbb{R}^n)}$.

We mention the two following examples for the diffusion ν satisfying A1.

Example 3.1. The case of the (truncated) log-normal fields, i.e.,

$$\nu(\omega, x) = \nu_0(x) + \exp(\sum_{j=1}^{M} z_j(\omega)\psi_j(x)), \tag{5}$$

where $\psi_j \in L^\infty(D;\mathbb{R})$ for $j = 0, \ldots, M$ and $\nu_0 \in L^\infty(D;\mathbb{R})$ with $\operatorname{ess\,inf}_{x\in D} \nu_0(x) = \underline{\nu} > 0$. The random variables $z_j : \Omega \to \mathbb{R}$ have zero means, they are pairwise uncorrelated, and they are truncated at some large enough lower and upper bounds, see e.g., [20, page 25] for more details. For every $z = (z_1, \ldots, z_M)$, the following quantities are assumed to be well defined.

$$\begin{aligned}\nu_{\max}(\omega) &= \operatorname*{ess\,sup}_{x\in D} \nu_0(x) + \exp(\sum_{j=1}^{M} |z_j(\omega)| \|\psi_j\|_{L^\infty(D;\mathbb{R})}),\\ \nu_{\min}(\omega) &= \operatorname*{ess\,inf}_{x\in D} \nu_0(x) + \exp(-\sum_{j=1}^{M} |z_j(\omega)| \|\psi_j\|_{L^\infty(D;\mathbb{R})}).\end{aligned} \tag{6}$$

Since the ranges of z_j for $j = 1, \ldots, M$ are bounded, we have (4) for numbers $\infty > \overline{\nu} \geq \underline{\nu} > 0$.

Example 3.2. We can also consider the coefficient ν to be characterized by a sequence of scalar random variables $\{z_j\}_{j\geq 1}$ with

$$\nu(\omega, x) = \nu_0(x) + \sum_{j=1}^{\infty} z_j(\omega)\psi_j(x), \tag{7}$$

where $\psi_j \in L^\infty(D;\mathbb{R})$ for $j \geq 1$, and $z_j : \Omega \to \mathbb{R}$ for $j = 1, 2, \cdots$ are independent random variables which are distributed identically and uniformly in $[-1, 1]$ such that the range of z_j is in $[-1, 1]$. Then all realizations of the random vector $z = (z_1, z_2, \ldots)$ are supported in the cube $[-1, 1]^{\mathbb{N}}$. Further, with $\nu^* := \operatorname{ess\,inf}_{x\in D} \nu_0(x)$ and some $\kappa > 0$, the functions ψ_j are supposed to satisfy

$$\sum_{j=1}^{\infty} \|\psi_j\|_{L^\infty(D;\mathbb{R})} \leq \frac{\kappa}{1+\kappa}\nu^*.$$

This assumption implies that the fluctuations (resp., deviations) from mean of the random coefficient $\nu(x, \omega)$ in (7) are dominated by the mean field, i.e., that they are small with respect to the deterministic mean field. Then, we have

$$\begin{aligned}\nu_{\max}(\omega) &= \operatorname*{ess\,sup}_{x\in D} \nu_0(x) + \sum_{j=1}^{\infty} |z_j(\omega)| \|\psi_j\|_{L^\infty(D;\mathbb{R})},\\ \nu_{\min}(\omega) &= \nu^* - \sum_{j=1}^{\infty} |z_j(\omega)| \|\psi_j\|_{L^\infty(D;\mathbb{R})},\end{aligned} \tag{8}$$

and the inequality in the right hand side of (4) holds with $\underline{\nu} := \nu^* - \frac{\kappa}{1+\kappa}\nu^* = \frac{1}{1+\kappa}\nu^*$.

3.1 Well-posedness of state equation

We start with the well-posedness of (CS). In this regard, we consider for $\omega \in \Omega$ a.s., the following auxiliary random linear parabolic equation

$$\begin{cases} \partial_t y - \nabla \cdot (\nu(\omega, x)\nabla y) + a(t,x)y + \nabla \cdot (b(t,x)y) = f(t,x,\omega) & (t,x) \in (t_0, t_0+T) \times D, \\ y = 0 & (t,x) \in (t_0, t_0+T) \times \partial D, \\ y(t_0) = y_0(\omega, x) & x \in D, \end{cases} \tag{9}$$

and define the following notion of weak solution.

Definition 3.1. *Let $(T, t_0, y_0, f) \in \mathbb{R}^2 \times H_{\mathbb{P}} \times L^2((t_0, t_0+T); V'_{\mathbb{P}})$ be given. Then, a random field $y \in W_{\mathbb{P}}(t_0, t_0+T)$ is referred to as a weak solution of (9), if it satisfies*

$$\begin{aligned} &\int_{t_0}^{t_0+T} \langle \partial_t y(t), \varphi(t) \rangle_{V'_{\mathbb{P}}, V_{\mathbb{P}}} \, dt + \int_{t_0}^{t_0+T} \int_{\omega} \int_D \nu \nabla y(t) \cdot \nabla \varphi(t) \, dx \, d\mathbb{P}(\omega) \, dt \\ &+ \int_{t_0}^{t_0+T} \langle a(t) y(t), \varphi(t) \rangle_{V'_{\mathbb{P}}, V_{\mathbb{P}}} \, dt - \int_{t_0}^{t_0+T} \int_{\Omega} \int_D y(t) b(t) \cdot \nabla \varphi \, dx \, d\mathbb{P}(\omega) \, dt \\ &= \int_{t_0}^{t_0+T} \langle f(t), \varphi \rangle_{V'_{\mathbb{P}}, V_{\mathbb{P}}} \, dt \qquad \textit{for all } \varphi \in L^2((t_0, t_0+T); V_{\mathbb{P}}), \end{aligned} \tag{10}$$

and $y(t_0) = y_0$ is satisfied in $H_{\mathbb{P}}$. Here we use $\langle \cdot, \cdot \rangle_{V'_{\mathbb{P}}, V_{\mathbb{P}}} := \mathbb{E}\left[\langle \cdot, \cdot \rangle_{V', V}\right]$.

In the following we present the existence result and various a-priori estimates for the weak solution of (9). These estimates will be used frequently in the sequel.

Theorem 3.1. *For every multiple $(T, t_0, y_0, f) \in \mathbb{R}^2_{\geq 0} \times H_{\mathbb{P}} \times L^2((t_0, t_0+T); V'_{\mathbb{P}})$ equation (9) admits a unique weak random field $y \in W_{\mathbb{P}}(t_0, t_0+T)$ satisfying the following estimates*

$$\|y\|^2_{C([t_0, t_0+T]; H_{\mathbb{P}})} + \|y\|^2_{W_{\mathbb{P}}(t_0, t_0+T)} \leq c_1 \left(\|y_0\|^2_{H_{\mathbb{P}}} + \|f\|^2_{L^2((t_0, t_0+T); V'_{\mathbb{P}})} \right), \tag{11}$$

with c_1 depending on $(T, \overline{\nu}, \underline{\nu}, a, b, D)$. Moreover, we have the following observability inequality

$$\|y_0\|^2_{H_{\mathbb{P}}} \leq c_2 \left(1 + T^{-1} + \mathcal{N}(a,b)\right) \|y\|^2_{L^2((t_0, t_0+T); V_{\mathbb{P}})} + \|f\|^2_{L^2((t_0, t_0+T); V'_{\mathbb{P}})}, \tag{12}$$

with c_2 depending only on $(T, \overline{\nu}, \underline{\nu}, D)$.

Proof. From (4) and Assumption A2 it follows that the sesquilinear form

$$\begin{aligned} b(t, \psi, \varphi) = &\int_{\Omega} \int_D \nu \nabla \psi \cdot \nabla \varphi \, dx \, d\mathbb{P}(\omega) \, dt + \langle a(t) y(t), \varphi(t) \rangle_{V'_{\mathbb{P}}, V_{\mathbb{P}}} \\ &- \int_{\Omega} \int_D y(t) b(t) \cdot \nabla \varphi \, dx \, d\mathbb{P}(\omega) \qquad \forall \psi, \varphi \in V_{\mathbb{P}}, \end{aligned} \tag{13}$$

is coercive and continuous. Thus there exist positive constants $c_{\min}$, $c_{\max}$, and c_0, such that for every $\psi, \varphi \in V_{\mathbb{P}}$ and a.e. $t \in (t_0, t_0+T)$ we have

$$|b(t, \psi, \varphi)| \leq c_{\max} \|\psi\|_{V_{\mathbb{P}}} \|\varphi\|_{V_{\mathbb{P}}}, \quad \text{and} \quad b(t, \psi, \psi) \geq c_{\min} \|\psi\|^2_{V_{\mathbb{P}}} - c_0 \|\psi\|^2_{H_{\mathbb{P}}}. \tag{14}$$

The rest of proof follows by using a Galerkin approximation with orthonormal basis functions $\{\psi_i \otimes \phi_j\}_{i,j \geq 1} \subset L^2_{\mathbb{P}}(\Omega; \mathbb{R}) \otimes V \cong V_{\mathbb{P}}$ and passing to the limit in the weak formations (10), where $\{\psi_i\}_{i, \geq 1}$ and $\{\phi_j\}_{j \geq 1}$ are orthonormal bases for the spaces $L^2_{\mathbb{P}}(\Omega; \mathbb{R})$ and V, respectively. The energy estimate (11) and (12) are obtained with the same arguments as in [1, Proposition 3.2.].

□

3.2 Stabilizability of the controlled system

In this section, we study the stabilizability of (CS) with respect to the number of actuators. Here we restrict ourselves to the rectangular domain, that is $D = [0, L_d]^d \subset \mathbb{R}^d$ and follow the arguments given in [25, Theorem 4.1] and [24].

It is well-known that the Laplacian operator $-\Delta : H^2(D;\mathbb{R}) \cap V \subset H \to H$ has a compact inverse and, thus, there exists a nondecreasing system of (repeated accordingly to their multiplicity) eigenvalues $\{\alpha_i\}_{i\geq 1}$ with its associated complete basis satisfying

$$0 < \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_i \to \infty \text{ with } -\Delta e_i = \alpha_i e_i.$$

For any $d \geq 1$, we construct, by induction, a family of pairs $(\mathcal{O}_N, \mathcal{E}_N)$ such that $H = \mathcal{O}_N \oplus \mathcal{E}_N^{\perp}$ for $N_\sigma = \sigma(N) := N^d$.

We start with the one-dimensional case, i.e., $d = 1$. For this case $D = (0, L_1)$ with $L_1 > 0$, and it has already been shown in [23, Lems. 4.3 and 5.1] that $L^2(D;\mathbb{R}) = \mathcal{O}_N \oplus \mathcal{E}_N^{\perp}$, if we take $N_\sigma = \sigma(N) := N$, and for a fixed $r \in (0, 1)$ define the following sets

$$\begin{aligned}
&\mathcal{E}_N = \mathcal{E}_N^{[1]} := \operatorname{span}\left\{e_{i,N}^{[1]} : i \in \mathbf{N}\right\} \subset H_0^1((0, L_1);\mathbb{R})\\
&\mathcal{O}_N = \mathcal{O}_N^{[1]} := \operatorname{span}\left\{\mathbf{1}_{O_{i,N}^{[1]}} : i \in \mathbf{N}\right\} \subset L^2((0, L_1);\mathbb{R})\\
&O_{i,N}^{[1]} := \left(c_{i,N}^{[1]} - \frac{rL_1}{2N}, c_{i,N}^{[1]} + \frac{rL_1}{2N}\right), \quad c_{i,N}^{[1]} := \frac{(2i-1)L_1}{2N}, \quad i \in \mathbf{N},
\end{aligned}$$

where $\mathbf{N} := \{1, 2, 3, \ldots, N\}$. Further, for $i \in \mathbf{N}$, $\mathbf{1}_{O_{i,N}^{[1]}}$ denote the indicator functions with supports $O_{i,N}^{[1]}$ and $e_{i,N}^{[1]}$ are the first eigenfunctions of the Laplacian in $L^2((0, L_1);\mathbb{R})$ under homogeneous Dirichlet boundary conditions.

Now, we deal with higher-dimensional rectangular domains $D = \bigtimes_{n=1}^{d}(0, L_n)$. Following the results in [15, sect. 4.8.1], it can be shown that the direct sum $L^2(D;\mathbb{R}) = \mathcal{E}_N \oplus \mathcal{O}_N^{\perp}$ property (note that $\mathcal{E}_N \oplus \mathcal{O}_N^{\perp} = \mathcal{O}_N \oplus \mathcal{E}_N^{\perp}$) holds also true for the choice $N_\sigma = \sigma(N) := N^d$ and the following setting

$$\begin{aligned}
&\mathcal{E}_N := \operatorname{span}\left\{e_{\mathbf{i},N}^{\times}(x) = \bigtimes_{n=1}^{d} e_{i_n,S}^{[n]}(x_n) : \mathbf{i} := (i_1, \ldots, i_d) \in \mathbf{N}^d\right\} \subset V\\
&\mathcal{O}_N := \operatorname{span}\left\{\mathbf{1}_{o_{\mathbf{i},N}^{\times}}(x) = \bigtimes_{n=1}^{d} \mathbf{1}_{O_{i_n,N}^{[n]}}(x_n) : \mathbf{i} \in \mathbf{N}^d\right\} \subset H\\
&O_{i,N}^{[n]} = \left(c_{i,N}^{[n]} - \frac{rL_n}{2N}, c_{i,N}^{[n]} + \frac{rL_n}{2N}\right), \quad c_{i,N}^{[n]} = \frac{(2i-1)L_n}{2N}, \quad i \in \mathbf{N},\\
&O_{\mathbf{i},N}^{\times}(x) = \bigtimes_{n=1}^{d} O_{\mathbf{i}_n,N}^{[n]}(x_n), \quad \mathbf{i} \in \mathbf{N}^d,
\end{aligned}$$

where $e_{i,N}^{[n]}$ with $i \in \mathbf{N}$ are the first eigenfunctions of the Laplacian in $L^2((0, L_n);\mathbb{R})$, $\mathbf{i} := (i_1, \ldots, i_d) \in \mathbf{N}^d$, and $x = (x_1, x_2, \ldots, x_d) \in \bigtimes_{n=1}^{d}(0, L_n)$. Moreover, for this choice of the pair $(\mathcal{O}_N, \mathcal{E}_N)$, it can be proven, with the same arguments as in [24, Section 5], that for every $N \in \mathbb{N}_0$ we have $H = \mathcal{E}_N \oplus \mathcal{O}_N^{\perp}$, and

$$\beta_N \geq c_\beta N^2 \quad \text{with} \quad \beta_N := \inf_{Q \in (V \cap \mathcal{O}_N^{\perp}) \setminus \{0\}} \frac{\|Q\|_V^2}{\|Q\|_H^2}, \tag{15}$$

where the constant c_β is independent of N and Q. Figure 1 illustrates the supports of actuators for the case $d = 2$ and different choices of N.

We have the following characterization from [23, Lemma 3.8] for the adjoint of the oblique projection.

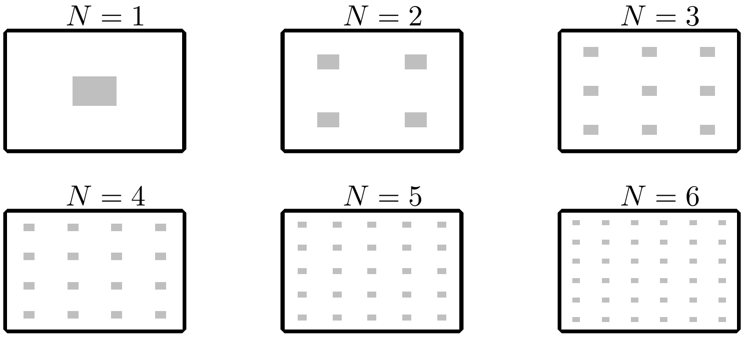

Figure 1: The actuators supports for $d=2$.

Lemma 3.1. *Suppose that F and G are closed subspaces of H, for which $H=F\oplus G$ holds. Then for the adjoint operator of $P_F^{G^\perp}\in\mathcal{L}(H)$, we have $(P_F^{G^\perp})^*=P_G^{F^\perp}$.*

In the next theorem, we investigate the stabilizability of the following control system

$$\begin{cases}\partial_t y-\nabla\cdot(\nu(\omega)\nabla y)+a(t,x)y+\nabla\cdot(b(t)y)=\sum_{i=1}^N u_i(t)\mathbf{1}_{O_i} & (t,x)\in(t_0,\infty)\times D,\\ y=0 & (t,x)\in(t_0,\infty)\times\partial D,\\ y(t_0)=y_0(\omega) & x\in D,\end{cases}\tag{16}$$

for almost surly $\omega\in\Omega$.

Theorem 3.2 (Uniform stabilizability of (16))**.** *For each $\mu>0$, there exists an integer $N^*\in\mathbb{N}_0$ such that for every $N\geq N^*$ there exists a feedback control vector $\bar{\mathbf{u}}(y)=(\bar u_1,\dots,\bar u_N)\in L^2((t_0,\infty);U_{\mathbb{P}}^N)\cong L^2_{\mathbb{P}}(\Omega;\mathbb{R})\otimes L^2((t_0,\infty);\mathbb{R}^N)$ for system (16) whose associated state satisfies*

$$\|y(t)\|^2_{H_{\mathbb{P}}}\leq e^{-\mu(t-t_0)}\|y_0\|^2_{H_{\mathbb{P}}}\quad \textit{for all } t>0,\tag{17}$$

for any given $(t_0,y_0)\in\mathbb{R}_{\geq 0}\times H_{\mathbb{P}}$.

Proof. We set as the feedback control law

$$\sum_{i=1}^N \bar u_i(t,\omega)\mathbf{1}_{O_i}:=-\lambda P_{\mathcal{O}_N}^{\mathcal{E}_N^\perp}\Delta P_{\mathcal{E}_N}^{\mathcal{O}_N^\perp}y(t,\omega)\quad \text{for a.e. } t>0 \text{ a.s. } \omega\in\Omega,\tag{18}$$

with

$$\bar{\mathbf{u}}(y)=(\bar u_1(y),\dots,\bar u_N(y))^t:=-\lambda\mathcal{I}P_{\mathcal{O}_N}^{\mathcal{E}_N^\perp}\Delta P_{\mathcal{E}_N}^{\mathcal{O}_N^\perp}y(t,\omega),\tag{19}$$

where $\mathcal{I}:\mathcal{O}_N\to\mathbb{R}^N$ stands for the canonical isomorphism, and the numbers $\lambda>0$ and $N\in\mathbb{N}_0$ are specified below. Inserting (18) in equation (16), multiplying with y, and integrating over D, we obtain for almost every $t>0$ and almost surely $\omega\in\Omega$ that

$$\begin{aligned}&\frac{d}{2dt}\|y(t,\omega)\|_H^2+(\nu(t,\omega)\nabla y(t,\omega),\nabla y(t,\omega))_H+\langle a(t)y(t,\omega),y(t,\omega)\rangle_{V',V}\\ &-\langle y(t,\omega),b(t)\cdot\nabla y(t,\omega)\rangle_{V',V}+\lambda\langle P_{\mathcal{O}_N}^{\mathcal{E}_N^\perp}\Delta P_{\mathcal{E}_N}^{\mathcal{O}_N^\perp}y(t,\omega),y(t,\omega)\rangle_{V,V'}=0\end{aligned}\tag{20}$$

From now on, we omit ω for simplicity, i.e. $y(t,\omega)=y(t)$. From (20) and using A1, it follows that

$$\begin{aligned}&\frac{d}{dt}\|y(t)\|_H^2\leq -2\nu_{\min}(\omega)\|y(t)\|_V^2+2|\langle a(t)y(t),y(t)\rangle_{V',V}|\\ &+2|(b(t)y(t),\nabla y(t))_H|+2\lambda\langle P_{\mathcal{O}_N}^{\mathcal{E}_N^\perp}\Delta P_{\mathcal{E}_N}^{\mathcal{O}_N^\perp}y(t,\omega),y(t,\omega)\rangle_{V',V}.\end{aligned}$$

We use the following decomposition

$$y = \theta + \varphi \quad \text{with } \theta := P_{\mathcal{E}_N}^{\mathcal{O}_N^\perp} y \quad \text{and} \quad \varphi := P_{\mathcal{O}_N^\perp}^{\mathcal{E}_N} y,$$

which is justified due to the definition of the oblique projection. Further, since $\theta \in \mathcal{E}_N \subset V$, then $\Delta\theta \in V'$. Thus, the operator $P_{\mathcal{O}_N}^{\mathcal{E}_N^\perp}$ can be considered as its unique linear extension to V'. That is $P_{\mathcal{O}_N}^{\mathcal{E}_N^\perp} \Delta\theta \in \mathcal{O}_N \subset H \subset V'$ and we have

$$\langle P_{\mathcal{O}_N}^{\mathcal{E}_N^\perp} \Delta\theta, w\rangle_{V',V} = \langle \Delta\theta, P_{\mathcal{E}_N}^{\mathcal{O}_N^\perp} w\rangle_{V',V} \qquad \text{for all } w \in V,$$

which is well-defined due the fact that $P_{\mathcal{E}_N}^{\mathcal{O}_N^\perp} w \in \mathcal{E}_N \subset V$. Thus, we can write

$$-\langle P_{\mathcal{O}_N}^{\mathcal{E}_N^\perp} \Delta\theta, y\rangle_{V',V} = -\langle \Delta\theta, \theta\rangle_{V',V} = \|\theta\|_V^2. \tag{21}$$

From (20) and (21), its follows by repeated use of Young's inequality that

$$\begin{aligned}
&\frac{d}{dt}\|y(t)\|_H^2 \\
&\leq -2\nu_{\min}(\omega)\|y(t)\|_V^2 + 2|\langle a(t)y(t), y(t)\rangle_{V',V}| + 2|(b(t)y(t), \nabla y(t))_H| - 2\lambda\|\theta(t)\|_V^2 \\
&\leq -2\nu_{\min}(\omega)\|y(t)\|_V^2 + 2c\mathcal{N}(a,b)\|y(t)\|_H\|y(t)\|_V - 2\lambda\|\theta(t)\|_V^2 \\
&\leq -\nu_{\min}(\omega)\|\theta(t)+\varphi(t)\|_V^2 + \frac{c^2\mathcal{N}^2(a,b)}{\nu_{\min}(\omega)}\|\theta(t)+\varphi(t)\|_H^2 - 2\lambda\|\theta(t)\|_V^2, \\
&\leq -\nu_{\min}(\omega)\left(\|\theta(t)\|_V^2 + \|\varphi(t)\|_V^2\right) + \nu_{\min}(\omega)\left(\kappa_1\|\theta(t)\|_V^2 + \frac{1}{\kappa_1}\|\varphi(t)\|_V^2\right) \\
&+ \frac{c^2\mathcal{N}^2(a,b)}{\nu_{\min}(\omega)}\left(\|\theta(t)\|_H^2 + \|\varphi(t)\|_H^2\right) + \frac{c^2\mathcal{N}^2(a,b)}{\nu_{\min}(\omega)}\left(\kappa_2\|\theta(t)\|_H^2 + \frac{1}{\kappa_2}\|\varphi(t)\|_H^2\right) - 2\lambda\|\theta(t)\|_V^2,
\end{aligned} \tag{22}$$

where c is a generic constant that depends only on D, and the numbers $\kappa_1 > 0$ and $\kappa_2 > 0$ can be chosen arbitrary. Setting $\kappa_1 = \kappa_2 = 2$ in the above inequality, we obtain

$$\begin{aligned}
&\frac{d}{dt}\|y(t)\|_H^2 \leq -\left(2\lambda - \nu_{\min}(\omega)\right)\|\theta(t)\|_V^2 + \frac{3c^2\mathcal{N}^2(a,b)}{\nu_{\min}(\omega)}\|\theta(t)\|_H^2 \\
&- \frac{\nu_{\min}(\omega)}{2}\|\varphi(t)\|_V^2 + \frac{3c^2\mathcal{N}^2(a,b)}{2\nu_{\min}(\omega)}\|\varphi(t)\|_H^2 \leq -\Theta_\theta(N,\lambda)\|\theta(t)\|_H^2 - \Theta_\varphi(N,\lambda)\|\varphi(t)\|_H^2,
\end{aligned} \tag{23}$$

where the constants Θ_θ and Θ_φ are defined by

$$\Theta_\theta(\omega, N, \lambda) := \left(2\lambda - \nu_{\min}(\omega)\right)\alpha_1 - \frac{3c^2\mathcal{N}^2(a,b)}{\nu_{\min}(\omega)}, \tag{24}$$

$$\Theta_\varphi(\omega, N, \lambda) := \frac{\nu_{\min}(\omega)}{2}\beta_N - \frac{3c^2\mathcal{N}^2(a,b)}{2\nu_{\min}(\omega)}, \tag{25}$$

with β_N given in (15) and α_1 as the smallest eigenvalue of the Laplacian with homogeneous Dirichlet boundary conditions.

Choosing N^* and λ^* such that

$$\beta_{N^*} \geq \frac{2}{\underline{\nu}}\left(4\mu + \frac{3c^2\mathcal{N}^2(a,b)}{2\underline{\nu}}\right) \qquad \text{and} \qquad \lambda^* \geq \frac{1}{2\alpha_1}\left(4\mu + \frac{3c^2\mathcal{N}^2(a,b)}{\underline{\nu}}\right) + \frac{\overline{\nu}}{2}. \tag{26}$$

We can infer for every $N \geq N^*$, $\lambda \geq \lambda^*$ that $\Theta_\theta(N,\lambda) \geq 4\mu$ and $\Theta_\varphi(N,\lambda) \geq 4\mu$. Therefore, together with (23) we obtain

$$\begin{aligned}
&\frac{d}{dt}\|y(t)\|_H^2 \leq -\Theta_\theta(N,\lambda)\|\theta(t)\|_H^2 - \Theta_\varphi(N,\lambda)\|\varphi(t)\|_H^2 \leq -4\mu\|\theta(t)\|_H^2 - 4\mu\|\varphi(t)\|_H^2 \\
&\leq -2\mu\left(\|\theta(t)\|_H^2 + \|\varphi(t)\|_H^2 + 2(\theta(t),\varphi(t))_H\right) \leq -2\mu\|\theta(t)+\varphi(t)\|_H^2 \leq -2\mu\|y(t)\|_H^2,
\end{aligned} \tag{27}$$

for a.e. $t > t_0$ and $\omega \in \Omega$ a.s.. Integrating (27) over interval (t_0, t) we obtain that

$$\|y(t,\omega)\|_H^2 \le e^{-2\mu(t-t_0)}\|y(t_0,\omega)\|_H^2 = e^{-2\mu(t-t_0)}\|y_0(\omega)\|_H^2 \quad \omega \in \Omega \text{ for a.s. } . \tag{28}$$

Finally, integrating (28) over Ω, we obtain

$$\|y(t)\|_{H_\mathbb{P}}^2 = \mathbb{E}\left[\|y(t)\|_H^2\right] \le e^{-2\mu(t-t_0)}\mathbb{E}\left[\|y(t_0)\|_H^2\right] = e^{-2\mu(t-t_0)}\|y_0\|_{H_\mathbb{P}}^2.$$

This together with the fact that $\bar{\mathbf{u}} \in L^2((t_0,\infty); U_\mathbb{P}^N)$ (see (19)) completes the proof. □

Remark 3.1. *Assume that $b = 0$ and $a \in L^\infty((0,\infty)\times D;\mathbb{R})$. Then, by replacing the term $c\mathcal{N}(a,b)\|y(t)\|_H\|y(t)\|_V$ with $\|a\|_{L^\infty((0,\infty)\times D;\mathbb{R})}\|y(t)\|_H^2$ in the third line of (22) and following the same lines of computations as above, (24) and (25) can be expressed as*

$$\Theta_\theta(\omega,N,\lambda) := 2\left(\lambda - \nu_{\min}(\omega)\right)\alpha_1 - 6\|a\|_{L^\infty((0,\infty)\times D;\mathbb{R})},$$
$$\Theta_\varphi(\omega,N,\lambda) := \nu_{\min}(\omega)\beta_N - 3\|a\|_{L^\infty((0,\infty)\times D;\mathbb{R})}.$$

Hence, for any given rate $\mu > 0$, the stabilizability result (17) holds for N^* and λ^* satisfying

$$\beta_{N^*} \ge \frac{1}{\underline{\nu}}\left(4\mu + 3\|a\|_{L^\infty((0,\infty)\times D;\mathbb{R})}\right) \quad \text{and} \quad \lambda^* \ge \frac{1}{\alpha_1}\left(2\mu + 3\|a\|_{L^\infty((0,\infty)\times D;\mathbb{R})}\right) + \overline{\nu}. \tag{29}$$

3.3 Stability of stochastic RHC

In this section, we investigate the stability of the receding horizon algorithm 1. Our theoretical results are expressed in terms of the finite- and infinite-horizon value functions and are based on the stability result given in the previous section.

We have the following stability result for the stochastic RHC $\mathbf{u}_{rh}$ obtained by Algorithm 1 with $U = U_\mathbb{P}^N$ and the choices of $\ell(t,y) = \|y\|_V^2$ and $\ell(t,y) = \|y\|_H^2$ for almost every $t \in (0,\infty)$.

Theorem 3.3. *Suppose that $D \subset \mathbb{R}^d$ with $d \ge 1$ is a rectangle. Then for every choice of $\ell(t,\cdot) = \|\cdot\|_V^2$ or $\ell(t,\cdot) = \|\cdot\|_H^2$, there exits an $N^* = N^*(a,b,\nu) \in \mathbb{N}$ such the RHC computed by Algorithm 1 with $U := U_\mathbb{P}^N$ is, for every $N \ge N^*$ and set of actuators $\mathbf{1}_{O_i}$ with $i = 1,\dots,N$ given in the previous section, suboptimal and stabilizing. That is for every given $\delta > 0$ there exist numbers $T^* > \delta$, and $\alpha \in (0,1)$, such that for every fixed prediction horizon $T \ge T^*$, and every $y_0 \in H_\mathbb{P}$ the control $\mathbf{u}_{rh} \in L^2((0,\infty); U_\mathbb{P}^N)$ provided Algorithm 1 by satisfies the suboptimality inequality*

$$\alpha V_\infty(y_0) \le \alpha J_\infty(\mathbf{u}_{rh}; 0, y_0) \le V_T(0,y_0) \le V_\infty(y_0). \tag{30}$$

Furthermore, we have

$$\|y(t)\|_{H_\mathbb{P}}^2 \to 0 \quad \text{as } t \to \infty, \tag{31}$$

for the choice of $\ell(t,\cdot) = \|\cdot\|_H^2$, and

$$\|y(t)\|_{H_\mathbb{P}}^2 \le e^{-\zeta t}c_e\|y_0\|_{H_\mathbb{P}}^2 \quad \text{for } t \ge 0, \tag{32}$$

for the choice of $\ell(t,\cdot) = \|\cdot\|_V^2$, where ζ and c_e are independent of y_0.

Proof. Algorithm 1 corresponds to the receding horizon framework introduced in [1] for time-varying linear evolution equations adapted to the spaces $V_\mathbb{P} \hookrightarrow H_\mathbb{P} \hookrightarrow V'_\mathbb{P}$. The stability of this framework is based on the three key conditions which we will verify here. The rest of the proof follows with the same arguments as given in [1, Theorem 2.6].

P1: For every positive number T, V_T is globally decrescent with respect to the $H_\mathbb{P}$-norm. That is, there exists a continuous, non-decreasing, and bounded function $\gamma_2 : \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ such that

$$V_T(\bar{t}_0,\bar{y}_0) \le \gamma_2(T)\|\bar{y}_0\|_{H_\mathbb{P}}^2 \quad \text{for all } (\bar{t}_0,\bar{y}_0) \in \mathbb{R}_{\ge 0} \times H_\mathbb{P}. \tag{33}$$

It is sufficient to chose $N^* \in \mathbb{N}$ as the smallest number for which

$$\beta_{N^*} > \frac{3c^2\mathcal{N}^2(a,b)}{\underline{\nu}^2}$$

holds. In this case, for almost surly $\omega \in \Omega$, we have

$$\beta_{N^*} > \frac{3c^2\mathcal{N}^2(a,b)}{\nu^2_{\min}(\omega)}, \tag{34}$$

and we can use Theorem 3.2 to verify the stabilizability. Indeed, setting $\bar{u} \in L^2((0,\infty);U^N_{\mathbb{P}})$ as in (18) for any $N \geq N^*$ and $\lambda \geq \lambda^*$ with

$$\lambda^* := \frac{1}{2\alpha_1}\left(4\mu + \frac{3c^2\mathcal{N}^2(a,b)}{\underline{\nu}}\right) + \frac{\overline{\nu}}{2}, \tag{35}$$

we obtain (17) for the rate

$$\mu := \frac{\underline{\nu}}{8}\left(\beta_{N^*} - \frac{3c^2\mathcal{N}^2(a,b)}{\underline{\nu}^2}\right).$$

Further, for this control we can write

$$\bar{\mathbf{u}}(\bar{y}(t,\omega)) = (\bar{u}_1(\bar{y}(t,\omega)),\ldots,\bar{u}_N(\bar{y}(t,\omega)))^T := -\lambda\mathcal{I}P^{\mathcal{E}_N^\perp}_{\mathcal{O}_N}\Delta P^{\mathcal{O}_N^\perp}_{\mathcal{E}_N}\bar{y}(t,\omega), \tag{36}$$

where $\mathcal{I} : \mathcal{O}_N \to \mathbb{R}^N$ denotes the canonical isomorphism. For the control $\bar{\mathbf{u}}$ and its associated state $\bar{y} = y(\bar{\mathbf{u}})$ it holds that

$$V_T(\bar{t}_0,\bar{y}_0) \leq \frac{1}{2}\int_{\bar{t}_0}^{\bar{t}_0+T} \mathbb{E}\left[\ell(t,\bar{y}(t))\right]dt + \frac{\beta}{2}\int_{\bar{t}_0}^{\bar{t}_0+T}\mathbb{E}\left[|\bar{\mathbf{u}}(t)|^2_{\ell_2}\right]dt,$$

and, depending on the choice of ℓ, we have the following cases:

First case $\ell(t,\cdot) = \|\cdot\|^2_H$: Using the fact that

$$\|P^{\mathcal{E}_N^\perp}_{\mathcal{O}_N}\Delta P^{\mathcal{O}_N^\perp}_{\mathcal{E}_N}\|_{\mathcal{L}(H)} \leq c_P \text{ and } \|\mathcal{I}\|_{\mathcal{L}(\mathcal{O}_N,\mathbb{R}^N)} \leq \hat{c},$$

for positive constants c_P and $\hat{c}$, we obtain with (36) that

$$\begin{aligned} V_T(\bar{t}_0,\bar{y}_0) &\leq \frac{1}{2}\int_{\bar{t}_0}^{\bar{t}_0+T}\left(\|\bar{y}(t)\|^2_{H_{\mathbb{P}}}dt + \lambda\beta\hat{c}^2c_P^2\|\bar{y}(t)\|^2_{H_{\mathbb{P}}}\right)dt \\ &\leq \frac{1+\lambda\beta\hat{c}^2c_P^2}{2\mu}\left(1-e^{-\mu T}\right)\|\bar{y}_0\|^2_{H_{\mathbb{P}}} =: \gamma_2(T)\|\bar{y}_0\|^2_{H_{\mathbb{P}}}. \end{aligned} \tag{37}$$

Second case $\ell(t,\cdot) = \|\cdot\|^2_V$: In this case, with standard energy estimates, we have for almost every $t \in (\bar{t}_0,\bar{t}_0+T)$ and almost surly $\omega \in \Omega$ that

$$\begin{aligned} \frac{d}{2dt}\|\bar{y}(t)\|^2_H + \underline{\nu}\|\bar{y}(t)\|^2_V &\leq c\mathcal{N}(a,b)\|\bar{y}(t)\|_H\|\bar{y}(t)\|_V + \|\sum_{i=1}^N \bar{u}(t,\omega)\mathbf{1}_{O_i}\|_H\|\bar{y}(t)\|_H \\ &\leq \frac{1}{2}c_5\|\bar{y}(t)\|^2_H + \frac{\underline{\nu}}{2}\|\bar{y}(t)\|^2_V + \frac{1}{2}|\bar{\mathbf{u}}(t)|^2_{\ell_2}, \end{aligned}$$

where $c_5 := \left(\frac{c^2}{\underline{\nu}}\mathcal{N}^2(a,b) + N\max_{1\leq i\leq N}\|\mathbf{1}_{O_i}\|^2_H\right)$. Integrating over Ω and $(\bar{t}_0,\bar{t}_0+T)$, together with (37), we obtain

$$\begin{aligned} \int_{\bar{t}_0}^{\bar{t}_0+T}\|\bar{y}(t)\|^2_{V_{\mathbb{P}}}dt &\leq \frac{1}{\underline{\nu}}\left(\|\bar{y}(\bar{t}_0)\|^2_{H_{\mathbb{P}}} + c_5\int_{\bar{t}_0}^{\bar{t}_0+T}\|\bar{y}(t)\|^2_{H_{\mathbb{P}}}dt + \int_{\bar{t}_0}^{\bar{t}_0+T}\mathbb{E}\left[|\bar{\mathbf{u}}(t)|^2_{\ell_2}\right]dt\right) \\ &\leq \frac{1}{\underline{\nu}}\left(1 + \frac{c_5+\lambda\hat{c}^2c_P^2}{\mu}(1-e^{-\mu T})\right)\|\bar{y}_0\|^2_{H_{\mathbb{P}}}. \end{aligned} \tag{38}$$

Finally, using (38), we have

$$
\begin{aligned}
V_T(\bar{t}_0, \bar{y}_0) &\le \frac{1}{2}\int_{\bar{t}_0}^{\bar{t}_0+T} \left(\|\bar{y}(t)\|^2_{V_{\mathbb{P}}}\, dt + \lambda\beta\hat{c}^2 c_P^2 \|\bar{y}(t)\|^2_{H_{\mathbb{P}}}\right)\, dt \\
&\le \frac{1}{2\underline{\nu}}\left(1 + \frac{c_5 + \lambda(1+\beta\underline{\nu})\hat{c}^2 c_P^2}{\mu}(1-e^{-\mu T})\right)\|\bar{y}_0\|^2_{H_{\mathbb{P}}} =: \gamma_2(T)\|\bar{y}_0\|^2_{H_{\mathbb{P}}}.
\end{aligned}
$$

P2: For every $(\bar{t}_0, \bar{y}_0) \in \mathbb{R}_0 \times H_{\mathbb{P}}$, every finite horizon optimal control problem of the form $OP_T(\bar{t}_0, \bar{y}_0)$ admits a solution:

The objective function $J_T(\mathbf{u}; \bar{t}_0, y_0)$ is strictly convex, coercive, and nonnegative. Hence it is weakly lower semi-continuous and existence of a unique minimizer to $OP_T(\bar{t}_0, \bar{y}_0)$ follows from the direct method in the calculus of variations, see e.g., [12, Theorem 1.43].

Since P1 and P2 hold, we are in the position that we can apply [1, Theorem 6.2] and thus (30) holds. It remains now to show that (31) and (32) are satisfied.

First, we deal with (32). This follows using the same arguments given in the second part of [1, Theorem 6.2] together with Property P3 stating:

P3: For every $T > 0$, V_T is uniformly positive with respect to the $H_{\mathbb{P}}$-norm. In other words, for every $T > 0$ there exists a constant $\gamma_1(T) > 0$ such that we have

$$
V_T(\bar{t}_0, \bar{y}_0) \ge \gamma_1(T)\|\bar{y}_0\|^2_{H_{\mathbb{P}}} \quad \text{for every } (\bar{t}_0, \bar{y}_0) \in \mathbb{R}_{\ge 0} \times H_{\mathbb{P}}. \tag{39}
$$

We will next verify this property. For any arbitrary given $(\bar{t}_0, \bar{y}_0) \in \mathbb{R}_{\ge 0} \times H_{\mathbb{P}}$ and control $\mathbf{u} \in L^2((\bar{t}_0, \bar{t}_0 + T); U^N_{\mathbb{P}})$, we have by (12) that

$$
\|\bar{y}_0\|^2_{H_{\mathbb{P}}} \le c_2(1 + T^{-1} + \mathcal{N}(a,b)) \int_{\bar{t}_0}^{\bar{t}_0+T} \|\bar{y}(t)\|^2_{V_{\mathbb{P}}} dt + \int_{\bar{t}_0}^{\bar{t}_0+T} \|\sum_{i=1}^N \bar{u}_i \mathbf{1}_{O_i}\|^2_{V'_{\mathbb{P}}} dt.
$$

Together with the estimate

$$
\begin{aligned}
\int_{\bar{t}_0}^{\bar{t}_0+T} \|\sum_{i=1}^N \bar{u}_i \mathbf{1}_{O_i}\|^2_{V'_{\mathbb{P}}} dt &\le i_{H_{\mathbb{P}}, V'_{\mathbb{P}}} \int_{\bar{t}_0}^{\bar{t}_0+T} \|\sum_{i=1}^N \bar{u}_i \mathbf{1}_{O_i}\|^2_{H_{\mathbb{P}}} dt \\
&\le i_{H_{\mathbb{P}}, V'_{\mathbb{P}}} N \max_{1\le i\le N} \|\mathbf{1}_{O_i}\|^2_H \int_{\bar{t}_0}^{\bar{t}_0+T} \mathbb{E}\left[|\bar{\mathbf{u}}(t)|^2_{\ell_2}\right]\, dt,
\end{aligned}
$$

we obtain (39) with $\gamma_1(T) := \left(\max\left\{2c_2(1+T^{-1}+\mathcal{N}(a,b)), \frac{2}{\beta}(i_{H_{\mathbb{P}}, V'_{\mathbb{P}}} N \max_{1\le i\le N}\|\mathbf{1}_{O_i}\|^2_H)\right\}\right)^{-1}$, where $i_{H_{\mathbb{P}}, V'_{\mathbb{P}}}$ is the embedding constant from $H_{\mathbb{P}}$ into $V'_{\mathbb{P}}$. Therefore P3 holds and this completes the verification of (32).

Next we prove that (31) holds. Using (30) and (33), we can write

$$
\int_0^\infty \|y_{rh}(t)\|^2_{H_{\mathbb{P}}}\, dt \le \frac{2\gamma_2(T)}{\alpha}\|y_0\|^2_{H_{\mathbb{P}}} \quad \text{and} \quad \int_0^\infty \mathbb{E}\left[|\mathbf{u}_{rh}(t)|^2_{\ell_2}\right]\, dt \le \frac{2\gamma_2(T)}{\alpha\beta}\|y_0\|^2_{H_{\mathbb{P}}}. \tag{40}
$$

Further, with standard energy estimate we have for every $t \ge t_0$ that

$$
\begin{aligned}
\|y_{rh}(t)\|^2_{H_{\mathbb{P}}} + \underline{\nu}\int_0^t \|y_{rh}(t)\|^2_{V_{\mathbb{P}}}\, dt &\le \|y_0\|^2_{H_{\mathbb{P}}} + \left(\frac{c^2}{\underline{\nu}}\mathcal{N}^2(a,b) + N\max_{1\le i\le N}\|\mathbf{1}_{O_i}\|^2_H\right)\int_0^\infty \|y_{rh}(t)\|^2_{H_{\mathbb{P}}}\, dt \\
&+ \int_0^\infty \mathbb{E}\left[|\mathbf{u}_{rh}(t)|^2_{\ell_2}\right]\, dt \le c_6\|y_0\|^2_{H_{\mathbb{P}}},
\end{aligned}
$$

where $c_6 := \left(1 + \frac{2(1+\beta c_5)\gamma_2(T)}{\alpha\beta}\right)$ with $c_5 := \left(\frac{c^2}{\underline{\nu}}\mathcal{N}^2(a,b) + N\max_{1\le i\le N}\|\mathbf{1}_{O_i}\|^2_H\right)$. Thus, we can conclude that

$$
\|y_{rh}\|^2_{L^\infty((0,\infty);H_{\mathbb{P}})} \le c_6\|y_0\|^2_{H_{\mathbb{P}}} \quad \text{and} \quad \int_0^\infty \|y_{rh}(t)\|^2_{V_{\mathbb{P}}} \le \frac{c_6}{\underline{\nu}}\|y_0\|^2_{H_{\mathbb{P}}}. \tag{41}
$$

Further, we have for every $t'' \geq t' \geq 0$ that

$$
\begin{aligned}
&\|y_{rh}(t'')\|^2_{H_{\mathbb{P}}} - \|y_{rh}(t')\|^2_{H_{\mathbb{P}}} = \int_{t'}^{t''} \frac{d}{dt}\|y_{rh}(t)\|^2_{H_{\mathbb{P}}} dt,\\
&= 2\int_{t'}^{t''} \mathbb{E}\left[\langle y_{rh}(t), \nu\Delta y_{rh}(t) - a(t)y_{rh}(t) - \nabla\cdot(b(t)y_{rh}(t)) + \sum_{i=1}^{N}(u_{rh})_i(t)\mathbf{1}_{O_i}\rangle_{V,V'}\right] dt,\\
&\leq -2\underline{\nu}\int_{t'}^{t''} \|y_{rh}(t)\|^2_{V_{\mathbb{P}}} dt\\
&+ 2\int_{t'}^{t''} \mathbb{E}\left[\langle -a(t)y_{rh}(t) - \nabla\cdot(b(t)y_{rh}(t)) + \sum_{i=1}^{N}(u_{rh})_i(t)\mathbf{1}_{O_i}, y_{rh}(t)\rangle_{V,V'}\right] dt,\\
&\leq 2\mathcal{N}(a,b)\int_{t'}^{t''} \|y_{rh}(t)\|_{V_{\mathbb{P}}}\|y_{rh}(t)\|_{H_{\mathbb{P}}} dt\\
&+ 2(N\max_{1\leq i\leq N}\|\mathbf{1}_{O_i}\|^2_H)^{\frac{1}{2}}\int_{t'}^{t''} \|\mathbf{u}_{rh}(t)\|_{L^2_{\mathbb{P}}(\Omega;\mathbb{R}^N)}\|y_{rh}(t)\|_{H_{\mathbb{P}}} dt\\
&\leq 2\mathcal{N}(a,b)\big(\int_{t'}^{t''} \|y_{rh}(t)\|^2_{V_{\mathbb{P}}} dt\big)^{\frac{1}{2}}\big(\int_{t'}^{t''} \|y_{rh}(t)\|^2_{H_{\mathbb{P}}} dt\big)^{\frac{1}{2}}\\
&+ 2(N\max_{1\leq i\leq N}\|\mathbf{1}_{O_i}\|^2_H)^{\frac{1}{2}}\big(\int_{t'}^{t''} \mathbb{E}\left[|\mathbf{u}_{rh}(t)|^2_{\ell_2}\right] dt\big)^{\frac{1}{2}}\big(\int_{t'}^{t''} \|y_{rh}(t)\|^2_{H_{\mathbb{P}}} dt\big)^{\frac{1}{2}} \leq c_7\|y_0\|^2_{H_{\mathbb{P}}}(t''-t')^{\frac{1}{2}},
\end{aligned}
\tag{42}
$$

where $c_7 := 2\left(\mathcal{N}(a,b)\underline{\nu}^{\frac{-1}{2}}c_6 + (N\max_{1\leq i\leq N}\|\mathbf{1}_{O_i}\|^2_H)^{\frac{1}{2}}c_6^{\frac{1}{2}}\left(\frac{2\gamma_2(T)}{\alpha\beta}\right)^{\frac{1}{2}}\right)$ and (40) and (41) were used in the last inequality.

The rest of proof follows the same lines as in the proof of [1, Theorem 6.4] based on (42) and (40). □

3.4 Failure Probability

In this section, we are concerned with the failure probability for the receding horizon framework. For a given number of actuators $\bar{N}$, we compute an upper bound for the probability of the case, in which the stabilizability of the stochastic RHC computed by Algorithm 1 with($U := U^N_{\mathbb{P}}$) and the deterministic variant of Algorithm 1 ([1, Algorithm 1.1]) with control $U := \mathbb{R}^N$ are not guaranteed.

Concretely, let $\bar{N} \in \mathbb{N}_0$ be a given number of actuators. Recalling the proof of Theorem 3.3, it can be seen that the condition P1 and, particularity, inequality (34) are essential. Therefore, P1 and the stabilizability of the controlled system may fail if

$$\nu^2_{\min}(\omega)\beta_{\bar{N}} \leq 3c^2\mathcal{N}^2(a,b). \tag{43}$$

In this case, for a given $y_0 \in H$, the existence of a stabilizing deterministic control $\mathbf{u}_{rh}(\omega) \in L^2((0,\infty);\mathbb{R}^N)$ with respect to H-norm which is suboptimal in the sense of (30) for

$$\min_{\mathbf{u}\in L^2((0,\infty);\mathbb{R}^{\bar{N}})} J^\omega_\infty(\mathbf{u};0,y_0) := \frac{1}{2}\int_0^\infty \ell(t,y(t))dt + \frac{\beta}{2}\int_0^\infty |\mathbf{u}(t)|^2_{\ell_2}dt, \tag{44}$$

and also, for any given $y_0 \in H_{\mathbb{P}}$ the existence a stabilizing stochastic control $\mathbf{u}_{rh} \in L^2((0,\infty);U^{\bar{N}}_{\mathbb{P}})$ with respect to $H_{\mathbb{P}}$-norm which is suboptimal for

$$\min_{\mathbf{u}\in L^2((0,\infty);U^{\bar{N}}_{\mathbb{P}})} J_\infty(\mathbf{u};0,y_0) = \frac{1}{2}\int_0^\infty \mathbb{E}\left[\ell(t,y(t))\right] dt + \frac{\beta}{2}\int_0^\infty \mathbb{E}\left[|\mathbf{u}(t)|^2_{\ell_2}\right] dt, \tag{45}$$

are not guaranteed for either of the choices of $\ell(t,y) = \|y\|^2_V$ and $\ell(t,y) = \|y\|^2_H$. Therefore, the failure probability, for both of the above problem formulations, can be expressed by

$$\mathbb{P}\left[\nu^2_{\min}(\omega)\beta_{\bar{N}} \leq 3c^2\mathcal{N}^2(a,b)\right] = \mathbb{P}\left[\nu_{\min}(\omega) \leq 3^{\frac{1}{2}}c\mathcal{N}(a,b)\beta_{\bar{N}}^{-\frac{1}{2}}\right]. \tag{46}$$

We consider now both Examples 3.1 and 3.2.

Example 3.1: Setting $\Gamma(\omega) := \sum_{j=1}^{M} |z_j(\omega)| \|\psi_j\|_{L^\infty(D)}$, we have

$$\mathbb{P}\left[\nu_{\min}(\omega) \leq 3^{\frac{1}{2}} c\mathcal{N}(a,b)\beta_{\bar{N}}^{-\frac{1}{2}}\right] = \mathbb{P}\left[\exp(-\Gamma(\omega)) \leq 3^{\frac{1}{2}} c\mathcal{N}(a,b)\beta_{\bar{N}}^{-\frac{1}{2}} - \underline{\nu}\right]. \tag{47}$$

where $3^{\frac{1}{2}} c\mathcal{N}(a,b)\beta_{\bar{N}}^{-\frac{1}{2}} - \underline{\nu} \geq 0$ since otherwise (43) is not valid. Using (15), (46), (47), and Markov's inequality, we get

$$\begin{aligned}
&\mathbb{P}\left[\nu_{\min}(\omega) \leq 3^{\frac{1}{2}} c\mathcal{N}(a,b)\beta_{\bar{N}}^{-\frac{1}{2}}\right] = \mathbb{P}\left[-\Gamma(\omega) \leq \log\left(3^{\frac{1}{2}} c\mathcal{N}(a,b)\beta_{\bar{N}}^{-\frac{1}{2}} - \underline{\nu}\right)\right] \\
&= \mathbb{P}\left[\Gamma(\omega) \geq \log\left(\frac{1}{3^{\frac{1}{2}} c\mathcal{N}(a,b)\beta_{\bar{N}}^{-\frac{1}{2}} - \underline{\nu}}\right)\right] \leq \mathbb{E}\left[e^{\Gamma}\right] e^{\log\left(3^{\frac{1}{2}} c\mathcal{N}(a,b)\beta_{\bar{N}}^{-\frac{1}{2}} - \underline{\nu}\right)} \\
&= \mathbb{E}\left[e^{\Gamma}\right]\left(3^{\frac{1}{2}} c\mathcal{N}(a,b)\beta_{\bar{N}}^{-\frac{1}{2}} - \underline{\nu}\right) \leq \mathbb{E}\left[e^{\Gamma}\right]\left(\left(\frac{3}{c_\beta}\right)^{\frac{1}{2}} c\mathcal{N}(a,b)\bar{N}^{-1} - \underline{\nu}\right).
\end{aligned}$$

where $\mathbb{E}\left[e^{\Gamma}\right]$ is bounded and without loss of generality, we assumed that $\beta_{\bar{N}}^{\frac{1}{2}}(1+\underline{\nu}) \geq 3^{\frac{1}{2}} c\mathcal{N}(a,b)$. Here we recall that $\beta_N \to \infty$ as $N \to \infty$.

Example 3.2: Setting $\Gamma(\omega) := \sum_{j=1}^{\infty} |z_j(\omega)| \|\psi_j\|_{L^\infty(D)}$, we obtain

$$\mathbb{P}\left[\nu_{\min}(\omega) \leq 3^{\frac{1}{2}} c\mathcal{N}(a,b)\beta_{\bar{N}}^{-\frac{1}{2}}\right] = \mathbb{P}\left[\Gamma(\omega) \geq \nu^* - 3^{\frac{1}{2}} c\mathcal{N}(a,b)\beta_{\bar{N}}^{-\frac{1}{2}}\right], \tag{48}$$

where, recalling that $\beta_N \to \infty$ as $N \to \infty$, we assume that $\nu^* - 3^{\frac{1}{2}} c\mathcal{N}(a,b)\beta_{\bar{N}}^{-\frac{1}{2}} > 0$. Using (15), (46), (48), and Markov's inequality, we obtain

$$\begin{aligned}
&\mathbb{P}\left[\nu_{\min}(\omega) \leq 3^{\frac{1}{2}} c\mathcal{N}(a,b)\beta_{\bar{N}}^{-\frac{1}{2}}\right] = \mathbb{P}\left[\Gamma(\omega) \geq \nu^* - 3^{\frac{1}{2}} c\mathcal{N}(a,b)\beta_{\bar{N}}^{-\frac{1}{2}}\right] \\
&\leq \frac{\mathbb{E}[\Gamma]}{\nu^* - 3^{\frac{1}{2}} c\mathcal{N}(a,b)\beta_{\bar{N}}^{-\frac{1}{2}}} \leq \frac{\sum_{j\geq 1} \mathbb{E}[|z_j|] \|\psi_j\|_{L^\infty(D)}}{\nu^* - (\frac{3}{c_\beta})^{\frac{1}{2}} c\mathcal{N}(a,b)\bar{N}^{-1}}.
\end{aligned}$$

4 Parabolic PDEs with log-normal diffusions

In this section, we study the case of log-normal diffusions defined by

$$\nu(\omega, x) = \exp(g(\omega, x)) \tag{49}$$

with g a Gaussian random field with zero mean. This class of diffusions is used in many applications, including those related to subsurface flow modeling and hydrology. More precisely, for each $x \in D$, $g(x, \cdot)$ is a Gaussian random variable, and thus $0 < \nu(\omega, x) < \infty$ for each $\omega \in \Omega$. However, for any $\epsilon > 0$ we have $\mathbb{P}[\nu(\cdot, x) > \epsilon^{-1}] > 0$, and thus its corresponding elliptic operator is not uniformly bounded from above over all possible realizations of ω. We also have $\mathbb{P}[\nu(\cdot, x) < \epsilon] > 0$, so the corresponding elliptic operator is not uniformly elliptic either.

Assumption 4.1. *Throughout this section, we impose the following conditions:*

A1: There are random variables $\nu_{\min}$, and $\nu_{\max}$, such that

$$0 < \nu_{\min}(\omega) \leq \nu(\omega, x) \leq \nu_{\max}(\omega) < \infty \quad \textit{for a.e. } x \in D \textit{ and a.s. } \omega \in \Omega, \tag{50}$$

where $\nu_{\max}(\omega)$, $\frac{1}{\nu_{\min}(\omega)} \in L^p_{\mathbb{P}}(\Omega; \mathbb{R})$ for $p \in [1, \infty)$.

A2: We also assume for a and b that

$$a \in L^\infty((0,\infty) \times D; \mathbb{R}) \text{ and } b = 0. \tag{51}$$

As an example for the diffusion constant with the log-normal distribution satisfying Assumption 4.1, we refer to [10, 13, 26].

Example 4.1. We set $g(\omega, x) := \sum_{j=1}^\infty z_j(\omega)\psi_j(x)$ in (49), where the functions $\psi_j \in L^\infty(D;\mathbb{R})$ for $j = 1, 2, \ldots$ are such that $\sum_{j=1}^\infty \|\psi_j(x)\|_{L^\infty(D;\mathbb{R})}$ is finite and the random variables z_j are i.i.d, standard normal random variables, that is, $z_j \sim \mathcal{N}(0,1)$ in $\mathbb{R}$. For describing the resulting random field we set $\mathcal{F} := \bigotimes_{j=1}^\infty \mathcal{B}(\mathbb{R})$, where $\mathcal{B}(\mathbb{R})$ is the Borel σ-algebra in $\mathbb{R}$. In this case, the probability measure can be expressed as the Gaussian product probability, that is, $\mathbb{P} := \bigotimes_{j=1}^\infty \mathcal{N}(0,1)$. For the well-posedness of ν in (49), we restrict z to be in the set $\Omega := \{z \in \mathbb{R}^\mathbb{N} : \sum_{j=1}^\infty |z_j| \|\psi_j\|_{L^\infty(D)} < \infty\}$. In this case, Ω is $\mathcal{F}$-measurable, and $\mathbb{P}(\Omega) = 1$ holds. Further, for every $\omega \in \Omega$, the following quantities are well-defined

$$\nu_{\max}(\omega) = \exp(\sum_{j=1}^\infty |z_j(\omega)| \|\psi_j\|_{L^\infty(D)}), \qquad \nu_{\min}(\omega) = \exp(-\sum_{j=1}^\infty |z_j(\omega)| \|\psi_j\|_{L^\infty(D)}), \tag{52}$$

and satisfy A1 in Assumptions 4.1. See [4, 26] for more details.

4.1 Well-posedness of state equation

For the log-normally distributed diffusion, the existence is more delicate. In fact, due to lack of integrability, we can not use the weak formulation (10) directly. In this case, first, we show that the solution $y(\omega) \in W(0,T)$ exists for $\omega \in \Omega$ a.s., then we justify the measurability and integrability of the mapping $y : \Omega \to W(0,T)$ with $\omega \mapsto y(\omega)$. The latter relies on controlling the integrability of all constants in the estimates.

Theorem 4.1. *Suppose that Assumption 4.1 holds. Then for every given $(t_0, T, y_0, f) \in \mathbb{R}^2_{>0} \times H \times L^2((t_0, t_0+T); H)$ equation (9) admits a unique solution $y \in W_\mathbb{P}(t_0, t_0+T)$ satisfying the following estimates*

$$\|y\|^2_{C([t_0,t_0+T];H_\mathbb{P})} + \|y\|^2_{W_\mathbb{P}(t_0,t_0+T)} \le c_3 \left(\|y_0\|_H^2 + \|f\|^2_{L^2((t_0,t_0+T);H)} \right), \tag{53}$$

with c_3 depending on (T, a, b, D, Ω). Moreover, we have the following observability inequality

$$\|y_0\|_H^2 \le c_4 \|y\|^2_{L^2((t_0,t_0+T);V_\mathbb{P})} + \|f\|^2_{L^2((t_0,t_0+T);H)}, \tag{54}$$

with c_4 depending only on (T, a, b, D, Ω).

Proof. The proof is mainly inspired by the arguments given in [10] for the well-posedness of the elliptic PDEs with log-normal diffusion. It follows the following steps: First, for each $\omega \in \Omega$ a.s., we consider the unique solution $y(\omega) \in W(0,T)$ and derive the estimates for the resulting family of solutions. Next, we show that $y(\omega)$ is measurable with respect to ω. Finally, we show the integrability of this solution. Throughout, c is a generic constant that is independent of ω.

Using standard arguments for deterministic parabolic PDEs, it can be shown that for $\omega \in \Omega$ a.s., (CS) admits a unique weak solution $y(\omega)$ with

$$\begin{aligned} \frac{d}{2dt}\|y(\omega,t)\|_H^2 + \nu_{\min}(\omega)\|y(\omega,t)\|_V^2 &\le \|a\|_{L^\infty((0,\infty)\times D;\mathbb{R})}\|y(\omega,t)\|_H^2 + (f(t), y(\omega,t))_H \\ &\le \frac{\left(1 + 2\|a\|_{L^\infty((0,\infty)\times D;\mathbb{R})}\right)}{2}\|y(\omega,t)\|_H^2 + \frac{1}{2}\|f(t)\|_H^2. \end{aligned} \tag{55}$$

Using Gronwall's inequality we obtain for every $t \in [t_0, t_0+T]$ that

$$\begin{aligned} &\|y(\omega,t)\|_H^2 + 2\nu_{\min}(\omega)\|y(\omega)\|^2_{L^2((t_0,t);V)} \\ &\qquad \le c \exp(T(1 + 2\|a\|_{L^\infty((0,\infty)\times D;\mathbb{R})})) \left(\|y_0\|_H^2 + \|f\|^2_{L^2((t_0,t_0+T);H)} \right). \end{aligned}$$

Therefore, we can infer that

$$\|y(\omega)\|^2_{L^2((t_0,t_0+T);V)} \leq \frac{c}{\nu_{\min}(\omega)} \left(\|y_0\|^2_H + \|f\|^2_{L^2((t_0,t_0+T);H)}\right). \tag{56}$$

Similarly, we can write

$$\begin{aligned}\|\partial_t y(\omega)\|^2_{L^2((t_0,t_0+T);V')} &\leq c\left(\nu^2_{\max}(\omega) + \|a\|^2_{L^\infty((0,\infty)\times D;\mathbb{R})}\right)\|y(\omega)\|^2_{L^2((t_0,t_0+T);V)} + c\|f\|^2_{L^2((t_0,t_0+T);H)} \\ &\leq \frac{c(1+\nu^2_{\max}(\omega))}{\nu_{\min}(\omega)}\left(\|y_0\|^2_H + \|f\|^2_{L^2((t_0,t_0+T);H)}\right)\end{aligned}$$

and thus, together with (56) we have

$$\|y(\omega)\|^2_{W(0,T)} \leq c_c(\omega)\left(\|y_0\|^2_H + \|f\|^2_{L^2(t_0,t_0+T;H)}\right), \tag{57}$$

where $c_c(\omega) := \frac{c(1+\nu^2_{\max}(\omega))}{\nu_{\min}(\omega)}$.

Next, we show the measurability of the solution operator. To do this, we show that y is a.s. the limit of a sequence of measurable functions. One can adapt the proof given in [10, Theorem 3.4]. Hence, we just sketch its main idea here. We first define, for every $n \in \mathbb{N}$, the sequence $\Omega_n \subset \Omega$ for which the diffusion coefficient is uniformly bounded

$$\Omega_n := \{\omega \in \Omega : \nu_{\max}(\omega) < n \text{ and } \nu_{\min}(\omega) > \frac{1}{n}\} \subset \Omega. \tag{58}$$

Then $\{\Omega_n\}_n$ is increasing with $\Omega = \cup_{n\in\mathbb{N}}\Omega_n$. Further, invoking Theorem 3.1, it follows that (CS) has the unique solution $y_n \in L^2_{\mathbb{P}|_{\Omega_n}}(\Omega; W(t_0, t_0+T))$ where $\mathbb{P}|_{\Omega_n}$ stands for the restriction of $\mathbb{P}$ to Ω_n. This solution can be extended by zero to the solution defined on the whole of Ω, which we denote again by y_n. By definition, y_n is measurable. Further, due to the fact that $\Omega = \cup_{n\in\mathbb{N}}\Omega_n$, that y_n solves (CS) for $\omega \in \Omega_n$ a.s., and by uniqueness of the solution, the function y is the a.s. limit of measurable functions $\{y_n\}_n$, and thus it is also measurable.

Now, we are in a position where we can show the integrability of the solution. Using A1 in Assumption 4.1, we integrate over Ω both of sides of (57), and conclude that

$$\|y\|^2_{L^p(\Omega;W(0,T))} \leq \tilde{c}(p)\left(\|y_0\|^2_H + \|f\|^2_{L^2((t_0,t_0+T);H)}\right),$$

for every $p \in [1,\infty)$, and a constant $\tilde{c}(p) > 0$ depending on p, using that $c_c(\omega)$ is p-integrable. In particular, due to (3), inequality (53) holds for $p = 2$ with a constant $c_3 := \tilde{c}(2) > 0$.

We next verify the observability estimate (54). Multiplying (9) by $\frac{T+t_0-t}{T}y(t)$, integrating over (t_0, t_0+T), and using Young's inequality, we obtain for $\omega \in \Omega$ a.s.

$$\begin{aligned}&\|y_0\|^2_H = \frac{1}{T}\int_{t_0}^{t_0+T} \|y(\omega,t)\|^2_H\,dt \\ &+ 2\int_{t_0}^{t_0+T} \frac{t_0+T-t}{T}\Big(\nu_{\max}(\omega)\|y(\omega,t)\|^2_V + \langle a(t)y(\omega,t), y(\omega,t)\rangle_{V',V} - \langle f(t), y(\omega,t)\rangle_{V',V}\Big)dt \\ &\leq c\left(T^{-1} + \nu_{\max}(\omega) + 1 + \|a\|_{L^\infty((0,\infty)\times D;\mathbb{R})}\right)\int_{t_0}^{t_0+T} \|y(\omega,t)\|^2_V\,dt + \int_{t_0}^{t_0+T} \|f(t)\|^2_H\,dt,\end{aligned} \tag{59}$$

where $c > 0$ depends only on D. Setting $c_o(\omega) := c\left(T^{-1} + \nu_{\max}(\omega) + 1 + \|a\|_{L^\infty((0,\infty)\times D;\mathbb{R})}\right)$, dividing (59) by $c_o(\omega)$, and integrating over Ω, we obtain (54) and the proof is complete. □

4.2 Stability of deterministic RHC

According to Theorem 4.1, the well-posedness of the state in $W_{\mathbb{P}}(0,T)$ for the log-normal diffusion, is justified only for deterministic initial and forcing functions. Due to the lack of integrability (see (57)), it is not clear how it can be extended for the random fields as initial and forcing functions. Therefore, first, at any time instances t_i of the receding horizon framework, we turn the random fields $y_{rh}(t_j,\cdot,\cdot)$ to a deterministic initial function $\bar{y}_0(\cdot) = \left(\mathbb{E}\left[y_{rh}(t_j,\cdot)^2\right]\right)^{\frac{1}{2}}$ by computing average of the squared function with respect to ω. Then we plug this deterministic initial function into the online open-loop problem. In this regard, we have modified and changed Algorithm 1 to develop Algorithm 2. Further, we need to restrict ourselves here to a deterministic control and, thus, for every $\bar{y}_0 \in H$ we consider the following performance index

$$J_T(\mathbf{u};\bar{t}_0,\bar{y}_0) := \frac{1}{2}\int_{\bar{t}_0}^{\bar{t}_0+T} \mathbb{E}\left[\|y(t)\|_V^2\right]\, dt + \frac{\beta}{2}\int_{\bar{t}_0}^{\bar{t}_0+T} |\mathbf{u}(t)|_{\ell_2}^2 dt. \tag{60}$$

In the next theorem, we investigate the stability of the control obtained by Algorithm 2 for $U := \mathbb{R}^N$ and $\ell(t,y) := \|y\|_V^2$.

Algorithm 2 Robust RHC(δ,T) for the log-normal diffusion

Require: The sampling time δ, the prediction horizon $T \geq \delta$, and the initial state y_0
Ensure: The stability of the RHC $\mathbf{u}_{rh}$.

We proceed through the steps of Algorithm 1 except that Steps 1, 4, and 5 are replaced by:

1. Compute $\mathbb{E}\left[y_0^2(x)\right]$ for $x \in D$ and set $(\bar{t}_0,\bar{y}_0) := \left(0, \left(\mathbb{E}\left[y_0^2\right]\right)^{\frac{1}{2}}\right)$ and $y_{rh}(0) = y_0$;
4. Compute $\mathbb{E}\left[y_{rh}(\bar{t}_0+\delta,x)^2\right]$ of the state for any $x \in D$ at time $\bar{t}_0+\delta$;
5. Update: $(\bar{t}_0,\bar{y}_0) \leftarrow \left(\bar{t}_0+\delta, \left(\mathbb{E}\left[y_{rh}(\bar{t}_0+\delta)^2\right]\right)^{\frac{1}{2}}\right)$;

Theorem 4.2. *Suppose that Assumption 4.1 holds and*

$$\nu_{\min}(\omega) + \operatorname{ess\,inf}\{a(t,x) : (t,x) \in (0,\infty)\times D\} > 0, \text{ for } \omega \in \Omega \text{ a.s..} \tag{61}$$

Then, for $N \geq 1$ with the set of actuators $\{\mathbf{1}_{O_i}\}_{i=1}^N$ given in Section 3.2 , Algorithm 2 for $\ell(t,\cdot) := \|\cdot\|_V^2$ is suboptimal and stabilizing. That is, for every given $\delta > 0$, there exist numbers $T^* > \delta$, and $\alpha \in (0,1)$, such that for every fixed prediction horizon $T \geq T^*$, and every $y_0 \in H_{\mathbb{P}}$, the RHC $\mathbf{u}_{rh} \in L^2((0,\infty);\mathbb{R}^N)$ satisfies the suboptimality inequality (30) *and exponential stability result* (32).

Proof. The proof is similar to the one of Theorem 3.1 and is based on the arguments given in [1, Theorem 2.6]. To be more precise, we need again to verify the properties P1-P3 given in the proof of Theorem 3.1 with respect to the H-norm in place of the $H_{\mathbb{P}}$-norm. After verifying P1-P3, according to the construction (see Steps 1, 4, and 5) of Algorithm 2, it can be easily shown that (33) and (39) hold at every time instance $t_i = i\delta$ for $y_{rh}(t_i)$ with respect to the $H_{\mathbb{P}}$-norm, that is, $V_T(t_i, y_{rh}(t_i)) \leq \gamma_2(T)\|y_{rh}(t_i)\|_{H_{\mathbb{P}}}^2$ and $V_T(t_i, y_{rh}(t_i)) \geq \gamma_1(T)\|y_{rh}(t_i)\|_{H_{\mathbb{P}}}^2$ hold for every $i = 0,1,2,\ldots$. The rest of the proof can be completed along the routine of the proof of [1, Theorem 2.6]. Therefore, we will confine ourselves here to the justification of properties P1-P3.

To verify P1, we set $\bar{\mathbf{u}} := 0$ in (CS) and define $\hat{\nu}(\omega) := \nu_{\min}(\omega) + \operatorname{ess\,inf}\{a(t,x) : (t,x) \in (0,\infty)\times D\}$. Then using the standard energy estimates, we obtain for any $(\bar{t}_0, T) \in \mathbb{R}^2_{\geq 0}$ and $\bar{y}_0 \in H$ that

$$\frac{d}{2dt}\|\bar{y}(\omega,t)\|_H^2 + \hat{\nu}(\omega)\|\bar{y}(\omega,t)\|_H^2 \leq 0 \quad \text{for almost every } t \geq \bar{t}_0,$$

Thus, we have for $\omega \in \Omega$ a.s. that

$$\|\bar{y}(\omega,t)\|_H^2 \le e^{-2\hat{\nu}(\omega)(t-\bar{t}_0)}\|\bar{y}_0\|_H^2. \tag{62}$$

Integrating over $(\bar{t}_0, \bar{t}_0+T)$, we obtain

$$\|\bar{y}(\omega)\|^2_{L^2((\bar{t}_0,\bar{t}_0+T);H)} \le \left(\frac{1-e^{-2\hat{\nu}(\omega)T}}{2\hat{\nu}(\omega)}\right)\|\bar{y}_0\|_H^2 \le \frac{1}{2\hat{\nu}(\omega)}\|\bar{y}_0\|_H^2.$$

Thus, similarly to (55), we can write

$$\begin{aligned}\|\bar{y}(\omega)\|^2_{L^2((\bar{t}_0,\bar{t}_0+T);V)} &\le \frac{1}{2\nu_{\min}(\omega)}\|\bar{y}_0\|_H^2 + \frac{\|a\|_{L^\infty((0,\infty)\times D;\mathbb{R})}}{\nu_{\min}(\omega)}\|\bar{y}(\omega)\|^2_{L^2((\bar{t}_0,\bar{t}_0+T);H)}\\ &\le \left(\frac{1}{2\nu_{\min}(\omega)} + \frac{\|a\|_{L^\infty((0,\infty)\times D;\mathbb{R})}}{2\hat{\nu}(\omega)\nu_{\min}(\omega)}\right)\|\bar{y}_0\|_H^2.\end{aligned}$$

Integrating over $\omega \in \Omega$, we obtain

$$\|\bar{y}\|^2_{L^2((\bar{t}_0,\bar{t}_0+T);V_{\mathbb{P}}))} = \|\bar{y}\|^2_{L^2_{\mathbb{P}}(\Omega;L^2((\bar{t}_0,\bar{t}_0+T);V))} \le c_s\|\bar{y}_0\|_H^2, \tag{63}$$

where the integrability of $\frac{1}{\nu_{\min}(\omega)}$ and $\frac{1}{\hat{\nu}(\omega)\nu_{\min}(\omega)}$ is justified due to (61) and A1 in Assumption 4.1. Setting $\bar{\mathbf{u}} := 0$ in (60) and using (63), we arrive at

$$V_T(\bar{t}_0,\bar{y}_0) \le \frac{1}{2}\int_{\bar{t}_0}^{\bar{t}_0+T}\mathbb{E}\left[\|\bar{y}(t)\|_V^2\right]\,dt + \frac{\beta}{2}\int_{\bar{t}_0}^{\bar{t}_0+T}|\bar{\mathbf{u}}(t)|^2_{\ell_2}dt \le \frac{c_s}{2}\|\bar{y}_0\|_H^2,$$

for a positive constant $c_s > 0$. P2 and P3 follow with similar arguments as in the proof of Theorem 3.1 using the inequalities (53) and (54), respectively. □

Remark 4.1. *Condition* (61) *might be considered restrictive from the stabilizability point of view. However, while the uncontrolled system is stable, the exponential stability is not clear due to the lack of integrability (see* (62)*), since $\hat{\nu}$ is not uniformly bounded away from* 0*. Further, since $\nu_{\min}(\omega)$ can be really small and arbitrarily close to zero for some realization of $\omega \in \Omega$, the stability can be quite slow. However, using Algorithm 2 we are able to stabilize the system exponentially independent of all the perturbations of the dynamics caused by all possible realizations of the random variable ν.*

4.3 Failure probability

Similarly to Section 3.4, under Assumption 4.1, we derive an upper bound for the probability, where the stabilizability of receding horizon framework for both problems (44) and (45) is not guaranteed. Due to (29) in Remark 3.1, the stabilizability may fail if

$$\nu_{\min}(\omega)\beta_{\bar{N}} \le 3\|a\|_{L^\infty((0,\infty)\times D;\mathbb{R})},$$

holds. Thus, setting $\Gamma(\omega) := \sum_{j=1}^{\infty}|z_j(\omega)|\|\psi_j\|_{L^\infty(D)}$, we can write for the diffusion defined in Example 4.1 that

$$\begin{aligned}&\mathbb{P}\left[\nu_{\min}(\omega)\beta_{\bar{N}} \le 3\|a\|_{L^\infty((0,\infty)\times D;\mathbb{R})}\right] = \mathbb{P}\left[\nu_{\min}(\omega) \le 3\|a\|_{L^\infty((0,\infty)\times D;\mathbb{R})}\beta_{\bar{N}}^{-1}\right]\\ &\quad = \mathbb{P}\left[-\Gamma(\omega) \le \log\left(3\|a\|_{L^\infty((0,\infty)\times D;\mathbb{R})}\beta_{\bar{N}}^{-1}\right)\right] = \mathbb{P}\left[\Gamma(\omega) \ge \log\left(\frac{\beta_{\bar{N}}}{3\|a\|_{L^\infty((0,\infty)\times D;\mathbb{R})}}\right)\right].\end{aligned}$$

Further, for every $\kappa_0 > 0$, we can write

$$\mathbb{P}\left[\Gamma(\omega) \ge \log\left(\frac{\beta_{\bar{N}}}{3\|a\|_{L^\infty((0,\infty)\times D;\mathbb{R})}}\right)\right] \le \mathbb{E}\left[e^{\kappa_0\Gamma^2}\right]e^{-\kappa_0\log\left(\frac{\beta_{\bar{N}}}{3\|a\|_{L^\infty((0,\infty)\times D;\mathbb{R})}}\right)^2}. \tag{64}$$

where we have used the Markov inequality in the last step and we have assumed that $\beta_{\bar{N}} \geq 3\|a\|_{L^\infty((0,\infty)\times D;\mathbb{R})}$. Note that due to Fernique's theorem [8, Theorem 2.7], there exists $\kappa_0 > 0$ such that the expectation of the double exponential in the left-hand side of (64) is bounded. Then, we can conclude that

$$e^{-\kappa_0 \log\left(\frac{\beta_{\bar{N}}}{3\|a\|_{L^\infty((0,\infty)\times D;\mathbb{R})}}\right)^2} = \mathcal{O}(\beta_{\bar{N}}^{-p}(3\|a\|_{L^\infty((0,\infty)\times D;\mathbb{R})})^p) \tag{65}$$

for every $p \in [1,\infty)$. This follows from the observation that for every $\kappa > 0$ and $p, x \in [1,\infty)$ we can write

$$e^{\kappa \log(x)^2} \geq C_p x^p \Leftrightarrow \kappa \log(x)^2 \geq \log(C_p) + p\log(x) \quad \Leftrightarrow \kappa \log(x)^2 - p\log(x) \geq \log(C_p)$$

for a constant $C_p > 0$ depending only on p. Further, the function $f(y) = \kappa y^2 - py$ is lower bounded by $-p/(4\kappa)$, that is, $\min_{y\in\mathbb{R}} \kappa y^2 - py = -p^2/(4\kappa)$. Then, it follows that for every $p, x \in [1,\infty)$, we have $e^{\kappa \log(x)^2} \geq e^{-\frac{p^2}{4\kappa}} x^p$. Hence, the existence of C_p is justified by setting $C_p := e^{-\frac{p^2}{4\kappa}}$. Finally, using (15) and (65), we conclude that

$$\mathbb{P}\left[\nu_{\min}(\omega)\beta_{\bar{N}} \leq 3\|a\|_{L^\infty((0,\infty)\times D;\mathbb{R})}\right] = \mathcal{O}(\bar{N}^{-2p}(c_\beta^{-1} 3\|a\|_{L^\infty((0,\infty)\times D;\mathbb{R})})^p)$$

for every $p \in [1,\infty)$.

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