Variable-Length Source Dispersions Differ under Maximum and Average Error Criteria

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Abstract—**Variable-length compression without prefix-free constraints and with side-information available at both encoder and decoder is considered. Instead of requiring the code to be error-free, we allow for it to have a non-vanishing error probability. We derive one-shot bounds on the optimal average codeword length by proposing two new information quantities; namely, the conditional and unconditional ε-cutoff entropies. Using these one-shot bounds, we obtain the second-order asymptotics of the problem under two different formalisms—the average and maximum probabilities of error over the realization of the side-information. While the first-order terms in the asymptotic expansions for both formalisms are identical, we find that the source dispersion under the average error formalism is, in most cases, strictly smaller than its maximum error counterpart. Applications to a certain class of guessing problems, previously studied by Kuzuoka [*IEEE Trans. Inf. Theory*, vol. 66, no. 3, pp. 1674–1690, 2020], are also discussed.**

Index Terms—**Variable-length compression, Conditional lossless source coding, Second-order asymptotics, Source dispersion, Massey–Arıkan guessing problem allowing errors**

I. INTRODUCTION

IN this paper, we are concerned with the problem of variable-length compression without prefix-free constraints. In the simplest version of this problem, a source X is to be compressed to finite-length binary strings. The objective is to ensure that the average codeword length is minimized under the condition that the source code is one-to-one. One-to-one codes have been studied by several researchers (see [2] and references therein). Specifically, Wyner [3] and Alon and Orlitsky [4] derived the following upper and lower bounds:

$$H(X) - \log(H(X) + 1) - \log e \leq L^*(X) \leq H(X), \quad (1)$$

respectively, where log stands for the base-2 logarithm and $L^*(X)$ stands for the minimum average codeword length of the one-to-one codes for the source X. A direct consequence of these bounds is that for a stationary memoryless source $X^n = (X_1, \ldots, X_n)$, one has

$$L^*(X^n) = n H(X) + O(\log n) \qquad (\text{as } n \to \infty). \quad (2)$$

The above-mentioned studies and results assume that the code is not allowed to commit any error. In practical latency-constrained applications, occasional errors are often tolerable.

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Hence, it is worthwhile to study the counterparts to the above zero-error results when one allows the code to have a decoding error probability ε that is non-vanishing. Towards this end, Kostina, Polyanskiy, and Verdú [5] showed that the fundamental limit on the average codeword length $L^*(\varepsilon, X^n)$, again without the prefix-free constraint, admits the following asymptotic expansion:

$$L^*(\varepsilon, X^n) = n(1 - \varepsilon) H(X) - \sqrt{n V(X)} f_G(\varepsilon) + O(\log n) \quad (3)$$

as $n \to \infty$ for every $0 \leq \varepsilon \leq 1$. In this expression, the quantity $V(X)$ denotes the variance of the information density of the source X (often referred to as the varentropy [6]) and the map $f_G : [0, 1] \to [0, 1/\sqrt{2\pi}]$ is defined as

$$f_G(s) := \begin{cases} \varphi(\Phi^{-1}(s)) & \text{if } 0 < s < 1, \\ 0 & \text{if } s = 0 \text{ or } s = 1, \end{cases} \quad (4)$$

$$\varphi(t) := \frac{1}{\sqrt{2\pi}} e^{-t^2/2}; \quad (5)$$

and $\Phi^{-1} : (0, 1) \to \mathbb{R}$ denotes the inverse function of the Gaussian cumulative distribution function

$$\Phi(u) := \int_{-\infty}^{u} \varphi(t) \, dt. \quad (6)$$

This is the first instance of the second-order asymptotics [7]–[11] for variable-length compression. It is worth noting, for our subsequent considerations, that the first-order term in (3) is $(1 - \varepsilon) H(X)$ and so the strong converse property, in the sense of Wolfowitz [12], does not hold. Additionally, since the second-order term $\sqrt{V(X)} f_G(\varepsilon)$ in (3) is nonnegative for all $0 < \varepsilon < 1$, the fundamental limit $L^*(\varepsilon, X^n)$ for variable-length compression is always *smaller* than the first-order optimal coding rate $n(1 - \varepsilon) H(X)$. This is in contrast to, say, almost lossless fixed-length source coding [7], [8] in which if the tolerable error probability ε is less than $1/2$, the second-order term is positive, which means that the optimal code rate at a finite blocklength n is *larger* than the first-order term.

A. Main Contributions

In this paper, we extend this setting and result by considering the presence of side-information Y^n at both encoder and decoder. In this case, the notion of the error probability can take one of two different forms. One can consider the *maximum* error probability in which we would like[1]

$$\mathbb{P}\{X^n \neq \hat{X}^n \mid Y^n\} \leq \varepsilon \qquad (\text{a.s.}). \quad (7)$$

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[1]Strictly speaking, Eq. (7) means error probability constraints except on a null set. In other words, the error constraint is imposed on the *essential supremum* of the left-hand side of (7). However, we call it the *maximum* error probability as usual.

Said differently, we require that the reconstructed source $\hat{X}^n$ is equal to the original source sequence X^n with probability at least $1-\varepsilon$ almost surely with respect to the side-information Y^n. This is obviously a more stringent criterion than the *average* error probability criterion in which one simply requires that

$$\mathbb{P}\{X^n \neq \hat{X}^n\} \leq \varepsilon. \tag{8}$$

Here, the error probability is *averaged* over the realizations of Y^n. Clearly, the rate of compression under the maximum error criterion is at least as large as the average error criterion. In this paper, we quantify this gap precisely in terms of the second-order asymptotics, i.e., the analogue of the term scaling $\sqrt{n}$ in (3). We show that the first-order terms in the asymptotic expansions are identical and equal to $n(1-\varepsilon)H(X \mid Y)$, but the source dispersion for the maximum error case is smaller than that of its average error counterpart. That is, the backoff from $n(1-\varepsilon)H(X \mid Y)$ is smaller for the former, more stringent, case compared to the latter. In fact, the maximum (resp. average) error source dispersion is the conditional (resp. unconditional) information variance of the conditional information density. By the law of total variance, the conditional information variance is not larger than its unconditional counterpart. It is easy to show that the difference is non-zero for most sources. En route to proving our second-order results, we develop new and novel one-shot bounds for both these error probability formalisms. We introduce two new information measures, namely the *unconditional* and *conditional ε-cutoff entropies*; in the n-shot setting, these characterize the fundamental compression limits up to terms scaling as $\mathrm{O}(\log n)$. Finally, we discuss applications of the second-order asymptotic results to guessing problems with a "giving-up" policy; this class of problems was recently introduced by Kuzuoka [13].

B. Related Works

1) Prefix-Free Codes: The problem of variable-length compression allowing errors was initiated by Han [14] who considered the fundamental limits of prefix-free codes with vanishing error probability. Han [14] derived a general formula for the normalized average codeword length. A general formula allowing for non-vanishing error probabilities was derived by Koga and Yamamoto [15]. For a stationary memoryless source X^n on a finite alphabet $\mathcal{X}^n$, Koga and Yamamoto's general formula can be reduced to the first-order term $n(1-\varepsilon)H(X)$, which coincides with (3) up to a term scaling as $\mathrm{o}(n)$. In fact, as mentioned by Kuzuoka and Watanabe [16, Remark 2], the asymptotics of the prefix-free codes and the one-to-one codes are equal up to a constant factor.

2) Guessing Problems: One-to-one codes with side-information are essentially equivalent to strategies for guessing problems [17], [18] via Campbell's source coding problem [19] without prefix-free constraints (cf. [20]–[22]). Kuzuoka [13] generalized the guessing problem by allowing positive error probabilities. The guesser can also give up guessing at each stage; in this case, an error is declared. Kuzuoka [13] derived general formulas of both Campbell's source coding problems and guessing problems with non-vanishing error probability by introducing the conditional smooth Rényi entropy and by exploiting its properties.

3) Conditional Rate-Distortion Theory and State-Dependent Channels: A related topic to our present considerations is the conditional rate-distortion problem [23], [24]. Gray considered the problem of lossy compression with common side-information at both encoder and decoder. The duality between source coding and state-dependent channel coding problems with side-information available at both encoder and decoder have been characterized by Cover and Chiang [25] and Pradhan, Chou, and Ramchandran [26].

4) Variable-Length Slepian–Wolf Coding: He, Lastras-Montaño, Yang, Jagmohan, and Chen [27] investigated fixed- and variable-length Slepian–Wolf coding problems [28] with error probabilities that vanish but not exponentially fast. They derived the second-order coding rates and showed that variable-length Slepian–Wolf coding has a better second-order term compared to its fixed-length counterpart. These are characterized by some forms of the conditional and unconditional information variances, and the superiority of variable-length Slepian–Wolf coding is characterized by these differences. Variable-length Slepian–Wolf coding problems were also investigated by Kimura and Uyematsu [29] and Kuzuoka and Watanabe [16].

C. Paper Organization

This paper is organized as follows: The problem setting is formulated in Section II. Section II-A presents some definitions and notations of information measures for a correlated source (X, Y). The ε-cutoff entropies are defined in Section II-B. Section II-C introduces the variable-length conditional lossless source coding problems. The main results of this study are given in Section III. Specifically, the second-order asymptotics of variable-length compression under maximum and average error formalisms are stated in Theorems 1 and 2 of Section III-A, respectively. Our one-shot coding theorems are stated in Lemmas 2 and 6 of Sections III-B and III-C, respectively; those are used to prove Theorems 1 and 2, respectively. Applications of Theorems 1 and 2 to guessing problems with a "giving-up policy" are discussed in Section IV. Section V concludes this study.

II. PRELIMINARIES

A. Information Measures for Correlated Sources

Assume throughout that the underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is rich enough so that all random variables (r.v.'s) are well-defined on the space. Consider a countably infinite alphabet[2] $\mathcal{X} = \{1, 2, \dots\}$ and an abstract alphabet $\mathcal{Y}$. Let X be an $\mathcal{X}$-valued r.v. and Y a $\mathcal{Y}$-valued r.v. Then, the pair (X, Y) can be thought of as a correlated source pair.

In the conditional source coding, the second source Y plays the role of the side-information for the first source X. We now introduce several information quantities. Let $P_{X|Y}(x \mid Y)$ be a

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[2]In this study, assume that the σ-algebra on a countable alphabet is always the power set of the alphabet, as usual.

version of the conditional probability $\mathbb{P}\{X = x \mid Y\}$ for each $x \in \mathcal{X}$.[3] Denote by

$$\iota(X \mid Y) = \iota_{X|Y}(X \mid Y) := \log \frac{1}{P_{X|Y}(X \mid Y)} \qquad (9)$$

the conditional information density of X given Y. Define three $\sigma(Y)$-measurable information measures of X given Y as follows:

$$\mathcal{H}(X \mid Y) := \mathbb{E}[\iota(X \mid Y) \mid Y], \qquad (10)$$

$$\mathcal{V}(X \mid Y) := \mathbb{E}[(\iota(X \mid Y) - \mathcal{H}(X \mid Y))^2 \mid Y], \qquad (11)$$

$$\mathcal{T}(X \mid Y) := \mathbb{E}[|\iota(X \mid Y) - \mathcal{H}(X \mid Y)|^3 \mid Y], \qquad (12)$$

where $\mathbb{E}[Z \mid W]$ stands for the conditional expectation of a real-valued r.v. Z given a sub-σ-algebra $\sigma(W)$ generated by a r.v. W. When $\mathcal{Y}$ is countable, we analogously define

$$\mathcal{H}(X \mid y) := \sum_{x \in \mathcal{X}} P_{X|Y}(x \mid y) \log \frac{1}{P_{X|Y}(x \mid y)}, \qquad (13)$$

$$\mathcal{V}(X \mid y) := \sum_{x \in \mathcal{X}} P_{X|Y}(x \mid y) \left(\log \frac{1}{P_{X|Y}(x \mid y)} - \mathcal{H}(X \mid y) \right)^2, \qquad (14)$$

$$\mathcal{T}(X \mid y) := \sum_{x \in \mathcal{X}} P_{X|Y}(x \mid y) \left| \log \frac{1}{P_{X|Y}(x \mid y)} - \mathcal{H}(X \mid y) \right|^3, \qquad (15)$$

for each $y \in \mathcal{Y}$, provided that $P_Y(y) := \mathbb{P}\{Y = y\} > 0$, where $P_{X|Y}(x \mid y) := \mathbb{P}\{X = x \mid Y = y\}$ stands for the conditional probability given the event $\{Y = y\}$. Moreover, we define four information measures of X given Y as follows:

$$H(X \mid Y) := \mathbb{E}[\mathcal{H}(X \mid Y)], \qquad (16)$$

$$V_{\mathrm{c}}(X \mid Y) := \mathbb{E}[\mathcal{V}(X \mid Y)], \qquad (17)$$

$$V_{\mathrm{u}}(X \mid Y) := \mathbb{E}[(\iota(X \mid Y) - H(X \mid Y))^2], \qquad (18)$$

$$T_{\mathrm{u}}(X \mid Y) := \mathbb{E}[|\iota(X \mid Y) - H(X \mid Y)|^3], \qquad (19)$$

where $\mathbb{E}[Z]$ stands for the expectation of a real-valued r.v. Z.

Remark 1. *The somewhat unconventional notations $\mathcal{H}(X \mid Y)$, $\mathcal{V}(X \mid Y)$, and $\mathcal{T}(X \mid Y)$ defined in* (10)–(12), *respectively, are introduced to indicate that they are $\sigma(Y)$-measurable r.v.'s, i.e., they are not deterministic quantities like $H(X \mid Y)$, $V_{\mathrm{c}}(X \mid Y)$, $V_{\mathrm{u}}(X \mid Y)$, and $T_{\mathrm{u}}(X \mid Y)$. The notation $\mathcal{H}(X \mid Y)$ was also adopted in* [30].

The quantity $H(X \mid Y)$ is the well-known conditional Shannon entropy of X given Y. In this study, we respectively call $V_{\mathrm{c}}(X \mid Y)$ and $V_{\mathrm{u}}(X \mid Y)$ the *conditional* and *unconditional information variances*[4] of X given Y. It follows by the law of total variance that

$$V_{\mathrm{u}}(X \mid Y) = V_{\mathrm{c}}(X \mid Y) + \mathbb{E}[(\mathcal{H}(X \mid Y) - H(X \mid Y))^2]. \qquad (20)$$

Thus, the unconditional information variance $V_{\mathrm{u}}(X \mid Y)$ is larger than $V_{\mathrm{c}}(X \mid Y)$ by the term $\mathbb{E}[(\mathcal{H}(X \mid Y) - H(X \mid Y))^2]$, and these variances coincide if and only if $\mathcal{H}(X \mid Y)$ is almost surely constant.

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[3]Note that $P_{X|Y}(\cdot \mid Y)$ is a probability measure on $\mathcal{X}$ almost surely (a.s.) because the conditional probability is σ-additive.

[4]These terminologies are inspired by Polyanskiy's second-order asymptotic analysis in the channel coding problem [10, Equations (3.97)–(3.100)].

B. ε-Cutoff Entropies

Given a real-valued r.v. Z, define the (unconditional) ε-cutoff transformation action of Z by

$$\langle Z \rangle_{\varepsilon} := \begin{cases} Z & \text{if } Z < \eta, \\ B\,Z & \text{if } Z = \eta, \\ 0 & \text{if } Z > \eta, \end{cases} \qquad (21)$$

where B denotes a Bernoulli r.v. with parameter $(1 - \beta)$ in which the independence $B \perp\!\!\!\perp Z$ holds, and two real parameters $\eta \in \mathbb{R}$ and $0 \le \beta < 1$ are chosen so that

$$\mathbb{P}\{Z > \eta\} + \beta\,\mathbb{P}\{Z = \eta\} = \varepsilon. \qquad (22)$$

This is the same definition as [5, Equation (13)], and the notation $\langle Z \rangle_{\varepsilon}$ is consistent with that used in [5]. In addition, given a real-valued r.v. Z and an arbitrary r.v. W, define the *conditional ε-cutoff transformation action of Z given W* by

$$\langle Z \mid W \rangle_{\varepsilon} := \begin{cases} Z & \text{if } Z < \eta_W, \\ B_W\,Z & \text{if } Z = \eta_W, \\ 0 & \text{if } Z > \eta_W, \end{cases} \qquad (23)$$

where B_W denotes a Bernoulli r.v. in which the conditional independence $B_W \perp\!\!\!\perp Z \mid W$ holds and

$$\mathbb{P}\{B_W = 0 \mid W\} = \beta_W \qquad \text{(a.s.)}, \qquad (24)$$

and two $\sigma(W)$-measurable real-valued r.v.'s $\eta_W \in \mathbb{R}$ and $0 \le \beta_W < 1$ are chosen so that

$$\mathbb{P}\{Z > \eta_W \mid W\} + \beta_W\,\mathbb{P}\{Z = \eta_W \mid W\} = \varepsilon \qquad \text{(a.s.)}. \qquad (25)$$

In this paper, we call η and η_W the *cutoff points* of $\langle Z \rangle_{\varepsilon}$ and $\langle Z \mid W \rangle_{\varepsilon}$, respectively.

Using these cutoff operations, we now define the *unconditional* and *conditional ε-cutoff entropies* as follows:

$$\mathfrak{C}_{\mathrm{u}}^{\varepsilon}(X \mid Y) := \mathbb{E}[\langle \iota(X \mid Y) \rangle_{\varepsilon}], \qquad (26)$$

$$\mathfrak{C}_{\mathrm{c}}^{\varepsilon}(X \mid Y) := \mathbb{E}[\langle \iota(X \mid Y) \mid Y \rangle_{\varepsilon}], \qquad (27)$$

respectively. Note that these ε-cutoff entropies are not additive in general.

Finally, the following proposition gives some basic properties of the ε-cutoff transformation actions.

Proposition 1. *Let Z be a nonnegative-valued r.v., W a $\mathcal{W}$-valued r.v. with an abstract alphabet $\mathcal{W}$, and $0 \le \varepsilon \le 1$ a real number. Then, it holds that*

$$\mathbb{E}[\langle Z \rangle_{\varepsilon}] = \min_{\epsilon:\,\mathbb{E}[\epsilon(Z)] \le \varepsilon} \mathbb{E}[(1 - \epsilon(Z))\,Z], \qquad (28)$$

$$\mathbb{E}[\langle Z \mid W \rangle_{\varepsilon}] = \min_{\epsilon:\,\mathbb{E}[\epsilon(Z,W)|W] \le \varepsilon\,\text{(a.s.)}} \mathbb{E}[(1 - \epsilon(Z,W))\,Z], \quad (29)$$

where the minimization in (28) *(resp.* (29)*) is taken over the measurable maps $\epsilon : [0, \infty) \to [0, 1]$ (resp. the measurable maps $\epsilon : [0, \infty) \times \mathcal{W} \to [0, 1]$) satisfying $\mathbb{E}[\epsilon(Z)] \le \varepsilon$ (resp.*

$\mathbb{E}[\epsilon(Z, W) \mid W] \leq \varepsilon$ *a.s.). Moreover, the following identities hold:*

$$\mathbb{E}[\langle Z \rangle_\varepsilon] = (1 - \varepsilon)\, \mathbb{E}[Z]$$
$$- \int_\eta^\infty \mathbb{P}\{Z > t\}\, \mathrm{d}t - \varepsilon\,(\eta - \mathbb{E}[Z]), \quad (30)$$

$$\mathbb{E}[\langle Z \mid W \rangle_\varepsilon \mid W] = (1 - \varepsilon)\, \mathbb{E}[Z \mid W]$$
$$- \int_{\eta_W}^\infty \mathbb{P}\{Z > t \mid W\}\, \mathrm{d}t$$
$$- \varepsilon\,(\eta_W - \mathbb{E}[Z \mid W]) \quad \text{(a.s.)}, \quad (31)$$

where η and η_W are given in (22) and (25) respectively. In addition, the following inequality holds:

$$\mathbb{E}[\langle Z \rangle_\varepsilon] \leq \mathbb{E}[\langle Z \mid W \rangle_\varepsilon]. \quad (32)$$

Furthermore, given two nonnegative-valued r.v.'s Z_1 and Z_2, it holds that

$$\mathbb{P}\{Z_1 \leq t\} \leq \mathbb{P}\{Z_2 \leq t\} \ (\forall t > 0)$$
$$\implies \quad \mathbb{E}[\langle Z_1 \rangle_\varepsilon] \geq \mathbb{E}[\langle Z_2 \rangle_\varepsilon], \quad (33)$$
$$\mathbb{P}\{Z_1 \leq t \mid W\} \leq \mathbb{P}\{Z_2 \leq t \mid W\} \ (\text{a.s. } \forall t > 0)$$
$$\implies \quad \mathbb{E}[\langle Z_1 \mid W \rangle_\varepsilon \mid W] \geq \mathbb{E}[\langle Z_2 \mid W \rangle_\varepsilon \mid W] \ \text{(a.s.)}. \quad (34)$$

Proof of Proposition 1: See Appendix A. ∎

Note that (30) and (31) in Proposition 1 are useful in the subsequent second-order asymptotic analysis of the ε-cutoff entropies $\mathfrak{C}_{\mathrm{u}}^\varepsilon(X \mid Y)$ and $\mathfrak{C}_{\mathrm{c}}^\varepsilon(X \mid Y)$, respectively. The identities (28) and (29) will be used in the proofs of one-shot bounds stated in Lemmas 6 and 2, respectively, of Sections III-B and III-C, respectively. It follows from (32) of Proposition 1 that

$$\mathfrak{C}_{\mathrm{u}}^\varepsilon(X \mid Y) \leq \mathfrak{C}_{\mathrm{c}}^\varepsilon(X \mid Y). \quad (35)$$

C. Variable-Length Compression Under Two Error Criteria

Given an integer $n \geq 1$, denote by $(X_1, Y_1), (X_2, Y_2), \ldots, (X_n, Y_n)$ n i.i.d. copies of the source pair (X, Y). Then, we may think of (X^n, Y^n) as a sequence of outputs from the stationary memoryless correlated source (X, Y), where $X^n = (X_1, \ldots, X_n)$ and $Y^n = (Y_1, \ldots, Y_n)$. In this subsection, we formalize the variable-length conditional (almost) lossless source coding problems. Let $\{0, 1\}^*$ be the set of finite-length binary strings containing the empty string $\varnothing$. For each $n \geq 1$, consider two random maps $F_n : \mathcal{X}^n \times \mathcal{Y}^n \to \{0, 1\}^*$ and $G_n : \{0, 1\}^* \times \mathcal{Y}^n \to \mathcal{X}^n$ in which both $F_n(X^n, Y^n)$ and $G_n(F_n(X^n, Y^n), Y^n)$ are $\mathcal{F}$-measurable. Then, we call the pair (F_n, G_n) a *variable-length stochastic code* for the source X^n with side-information Y^n available at both encoder F_n and decoder G_n. We illustrate this compression scheme in Fig. 1.

Remark 2. *Another way to consider a variable-length stochastic code is to design a $\{0, 1\}^*$-valued r.v. B_n and an $\mathcal{X}^n$-valued r.v. $\hat{X}^n$ in which those probability laws are determined by versions of the conditional probabilities $\mathbb{P}\{B_n = \boldsymbol{b} \mid X^n, Y^n\}$ for $\boldsymbol{b} \in \{0, 1\}^*$ and $\mathbb{P}\{\hat{X}^n = \boldsymbol{x} \mid B_n, Y^n\}$ for $\boldsymbol{x} \in \mathcal{X}^n$, respectively. Kostina et al. [5] studied variable-length stochastic codes without side-information Y^n in this manner.*

Let $\ell : \{0, 1\}^* \to \mathbb{N} \cup \{0\}$ be the length function of a finite-length binary string; e.g., $\ell(\varnothing) = 0$, $\ell(0) = \ell(1) = 1$, $\ell(00) = \ell(01) = \ell(10) = \ell(11) = 2$, $\ell(000) = \ell(001) = \ell(010) = \ell(011) = \ell(100) = \ell(101) = \ell(110) = \ell(111) = 3$, and so on. Given a variable-length stochastic code (F_n, G_n), we are interested in the average codeword length $\mathbb{E}[\ell(F_n(X^n, Y^n))]$ to measure the efficiency of the data compressor for the source X^n with side-information Y^n.

Definition 1 (Maximum error criterion). *Let $n \geq 1$ be an integer, and $L \geq 0$ and $0 \leq \varepsilon \leq 1$ real numbers. Given a source X with side-information Y, an $(n, L, \varepsilon)_{\max}$-code is a variable-length stochastic code (F_n, G_n) satisfying*

$$\mathbb{E}[\ell(F_n(X^n, Y^n))] \leq L, \quad (36)$$
$$\mathbb{P}\{X^n \neq G_n(F_n(X^n, Y^n), Y^n) \mid Y^n\} \leq \varepsilon \quad \text{(a.s.)}. \quad (37)$$

Definition 2 (Average error criterion). *Let $n \geq 1$ be an integer, and $L \geq 0$ and $0 \leq \varepsilon \leq 1$ real numbers. Given a source X with side-information Y, an $(n, L, \varepsilon)_{\mathrm{avg}}$-code is a variable-length stochastic code (F_n, G_n) satisfying*

$$\mathbb{E}[\ell(F_n(X^n, Y^n))] \leq L, \quad (38)$$
$$\mathbb{P}\{X^n \neq G_n(F_n(X^n, Y^n), Y^n)\} \leq \varepsilon. \quad (39)$$

Given a probability of error $0 \leq \varepsilon \leq 1$, this study deals with the fundamental limits of the average codeword length under these error criteria. Specifically, we will investigate the two operational quantities $L_{\max}^*(n, \varepsilon, X, Y)$ and $L_{\mathrm{avg}}^*(n, \varepsilon, X, Y)$ defined in (40) and (41), respectively, and presented at the top of the next page.

III. SECOND-ORDER ASYMPTOTICS AND ONE-SHOT BOUNDS

A. Statements of Second-Order Asymptotic Results

Recall the definition of the function $f_{\mathrm{G}} : [0, 1] \to [0, 1/\sqrt{2\pi}]$ in (4).

Theorem 1 (Under maximum error criterion). *Suppose that the following two hypotheses hold:[5]*

(a) *$\mathcal{V}(X \mid Y)$ is bounded away from zero almost surely; and*
(b) *$\mathcal{T}(X \mid Y)$ is bounded away from infinity almost surely.*

Then, it holds that

$$L_{\max}^*(n, \varepsilon, X, Y) = n\,(1 - \varepsilon)\, H(X \mid Y)$$
$$- \sqrt{n\, V_{\mathrm{c}}(X \mid Y)}\, f_{\mathrm{G}}(\varepsilon) + \mathrm{O}(\log n) \quad (42)$$

as $n \to \infty$ for every $0 \leq \varepsilon \leq 1$.

Proof of Theorem 1: See Section III-B. ∎

Theorem 2 (Under average error criterion). *Suppose that $T_{\mathrm{u}}(X \mid Y)$ is finite. Then, it holds that*

$$L_{\mathrm{avg}}^*(n, \varepsilon, X, Y) = n\,(1 - \varepsilon)\, H(X \mid Y)$$
$$- \sqrt{n\, V_{\mathrm{u}}(X \mid Y)}\, f_{\mathrm{G}}(\varepsilon) + \mathrm{O}(\log n) \quad (43)$$

as $n \to \infty$ for every $0 \leq \varepsilon \leq 1$.

[5]We say that a real-valued r.v. Z is *bounded away from zero* (resp. *infinity*) *almost surely* if there exists a positive constant c satisfying $\mathbb{P}\{|Z| > c\} = 1$ (resp. $\mathbb{P}\{|Z| < c\} = 1$).
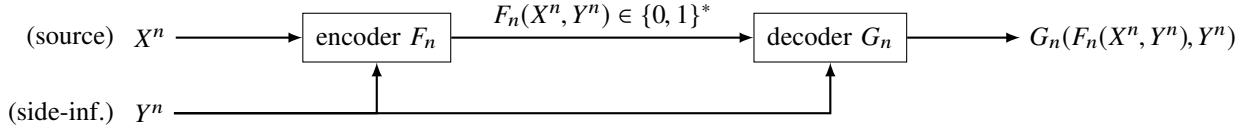
Fig. 1. Variable-length conditional source coding

$$L_{\max}^*(n, \varepsilon, X, Y) := \inf\{L > 0 \mid \text{there exists an } (n, L, \varepsilon)_{\max}\text{-code for the source } X \text{ with side-information } Y\} \quad (40)$$

$$L_{\mathrm{avg}}^*(n, \varepsilon, X, Y) := \inf\{L > 0 \mid \text{there exists an } (n, L, \varepsilon)_{\mathrm{avg}}\text{-code for the source } X \text{ with side-information } Y\} \quad (41)$$

Proof of Theorem 2: See Section III-C. ∎

Since an $(n, L, \varepsilon)_{\max}$-code is an $(n, L, \varepsilon)_{\mathrm{avg}}$-code, it is clear that

$$L_{\mathrm{avg}}^*(n, \varepsilon, X, Y) \leq L_{\max}^*(n, \varepsilon, X, Y). \quad (44)$$

Theorems 1 and 2 state that the first-order optimal coding rates are the same under both the maximum and average error criteria; they are equal to $n(1 - \varepsilon)H(X \mid Y)$. On the other hand, we see from (20) that unless $\mathcal{H}(X \mid Y)$ is almost surely constant, the optimal second-order coding rates differ under maximum and average error criteria. Since $f_{\mathrm{G}}(\varepsilon) \geq 0$ with equality if and only if either $\varepsilon = 0$ or $\varepsilon = 1$, note in Theorems 1 and 2 that a larger dispersion implies a shorter average codeword length on the $\sqrt{n}$ scale for every fixed $0 < \varepsilon < 1$. In particular, it follows from (20) that the variable-length source dispersion $V_{\mathrm{u}}(X \mid Y)$ under the average error criterion is larger than that $V_{\mathrm{c}}(X \mid Y)$ under the maximum error criterion by the term $\mathbb{E}[(\mathcal{H}(X \mid Y) - H(X \mid Y))^2]$.

Remark 3. *In channel coding, the counterparts of both conditional and unconditional information variances coincide for every capacity-achieving input distribution (cf. [10, Lemma 46] or [11, Lemma 62]). Since the first-order term determines the choice of input distribution (cf. [10, Lemma 48]), the ε-channel dispersion is determined by a capacity-achieving input distribution. Therefore, there is no difference between the conditional and unconditional information variances in channel coding in the absence of input cost constraints. On the other hand, the conditional and unconditional information variances are different for the problem at hand as there is no optimization over input distributions. Thus, the variable-length source dispersion under the maximum and average error formalisms are different.*

Remark 4. *In Theorems 1 and 2, the code is allowed to be stochastic. Namely, an encoder (resp. a decoder) outputs a compressed binary string B (resp. the reconstructed source $\hat{X}$) stochastically according to some probability law given a source X (resp. a compressed binary string B) and the side-information Y (see Remark 2). Since the average codeword length of a stochastic code is almost equal to that of a deterministic code up to a constant additive term of $(\log \mathrm{e})/\mathrm{e}$ (cf. [5, Section II-A]), our asymptotic analysis is the same as that if we assumed the code is deterministic. See also [13, Remark 6].*

In Appendix B, we show that even if the side-information alphabet $\mathcal{Y}$ is countably infinite, there is a correlated source (X, Y) that $\mathcal{V}(X \mid Y)$ and $\mathcal{T}(X \mid Y)$ are respectively not bounded away from zero and infinity a.s., but $T_{\mathrm{u}}(X \mid Y)$ is finite. Therefore, Hypotheses (a) and (b) in Theorem 1 are stronger than the hypothesis in Theorem 2 in general. On the other hand, these hypotheses are no longer needed if the alphabets of the source and the side-information are finite.

Proposition 2. *If X is supported on some finite subalphabet $\mathcal{A} \subset \mathcal{X}$, then Hypothesis (b) in Theorem 1 holds.*

Proposition 3. *If $\mathcal{Y}$ is finite, then (42) in Theorem 1 holds without Hypothesis (a).*

Proof of Propositions 2 and 3: See Appendix I. ∎

B. Proof of Theorem 1

Firstly, we establish one-shot bounds on the fundamental limit $L_{\max}^*(\varepsilon, X, Y) := L_{\max}^*(1, \varepsilon, X, Y)$; namely, we consider the case in which $n = 1$. The following lemma gives us a formula for $L_{\max}^*(\varepsilon, X, Y)$.

Lemma 1. *Given $0 \leq \varepsilon \leq 1$, it holds that*

$$L_{\max}^*(\varepsilon, X, Y) = \mathbb{E}[\langle \lfloor \log \varsigma_Y^{-1}(X) \rfloor \mid Y \rangle_\varepsilon], \quad (45)$$

where ς_Y stands for a random permutation on $\mathcal{X} := \{1, 2, \dots\}$ satisfying

$$P_{X|Y}(\varsigma_Y(1) \mid Y) \geq P_{X|Y}(\varsigma_Y(2) \mid Y) \geq \cdots \quad \text{(a.s.)}, \quad (46)$$

which rearranges the probability masses in $P_{X|Y}(\cdot \mid Y)$ in non-increasing order.

Proof of Lemma 1: See Appendix C. ∎

Lemma 1 tells us that in an optimal variable-length stochastic code (F^*, G^*) achieving (45), given the side-information Y, the set of source symbols that result in an error are those x with the smallest $P_{X|Y}(x \mid Y)$. Moreover, we see that the cutoff point η_Y of $\langle \lfloor \log \varsigma_Y^{-1}(X) \rfloor \mid Y \rangle_\varepsilon$ depends on the side-information Y, and the conditional overflow probability of this optimal code can be evaluated as

$$\mathbb{P}\{\ell(F^*(X, Y)) \leq \eta_Y \mid Y\} \geq 1 - \varepsilon \quad \text{(a.s.)}. \quad (47)$$

Using Lemma 1, we provide the following one-shot bounds on $L_{\max}^*(\varepsilon, X, Y)$ in terms of the conditional ε-cutoff entropy $\mathfrak{C}_{\mathrm{c}}^\varepsilon$ defined in (27).

Lemma 2. *For every $0 \leq \varepsilon \leq 1$, it holds that*

$$\mathfrak{C}_c^\varepsilon(X \mid Y) - \log(H(X \mid Y) + 1) - \log e$$
$$\leq L_{\max}^*(\varepsilon, X, Y) \leq \mathfrak{C}_c^\varepsilon(X \mid Y). \quad (48)$$

Proof of Lemma 2: See Appendix D. ∎

Note that Lemma 2 holds without Hypotheses (a) and (b) in Theorem 1. Since $H(X \mid Y) < \infty$ if Hypothesis (b) in Theorem 1 holds, and since $H(X^n \mid Y^n) = n H(X \mid Y)$ for each $n \geq 1$, Lemma 2 implies that

$$L_{\max}^*(n, \varepsilon, X, Y) = \mathfrak{C}_c^\varepsilon(X^n \mid Y^n) + \mathrm{O}(\log n) \quad (49)$$

as $n \to \infty$. Thus, it suffices to provide an appropriate asymptotic estimate on $\mathfrak{C}_c^\varepsilon(X^n \mid Y^n)$.

Lemma 3. *Given a fixed $0 \leq \varepsilon \leq 1$, it holds that*

$$\mathfrak{C}_c^\varepsilon(X^n \mid Y^n) = n(1 - \varepsilon) H(X \mid Y)$$
$$- \mathbb{E}\left[\sqrt{\mathcal{V}(X^n \mid Y^n)}\right] f_G(\varepsilon) + \mathrm{O}(1) \quad (50)$$

as $n \to \infty$, provided that Hypotheses (a) and (b) in Theorem 1 hold.

Proof of Lemma 3: See Appendix E. ∎

Unfortunately, obtaining an exact single-letter expression for the n-letter "dispersion" term $\mathbb{E}[\sqrt{\mathcal{V}(X^n \mid Y^n)}]$ that appears in (50) is difficult unless $\mathcal{V}(X \mid Y)$ is almost surely constant. In fact, it can be verified by Jensen's inequality that

$$\mathbb{E}\left[\sqrt{\mathcal{V}(X^n \mid Y^n)}\right] \leq \sqrt{n V_c(X \mid Y)}. \quad (51)$$

with equality if and only if $\mathcal{V}(X \mid Y)$ is almost surely constant, because $\mathcal{V}(X_1 \mid Y_1), \mathcal{V}(X_2 \mid Y_2), \ldots, \mathcal{V}(X_n \mid Y_n)$ will then be n i.i.d. copies of $\mathcal{V}(X \mid Y)$. Hence, Lemmas 2 and 3 can be readily reduced to Kostina *et al.*'s result [5, Theorem 4] in (3), provided that X and Y are independent.

The following lemma provides an asymptotic estimate of $\mathbb{E}[\sqrt{\mathcal{V}(X^n \mid Y^n)}]$.

Lemma 4. *If $\mathcal{V}(X \mid Y)$ is bounded away from infinity almost surely, then*

$$\mathbb{E}\left[\sqrt{\mathcal{V}(X^n \mid Y^n)}\right] = \sqrt{n V_c(X \mid Y)} + \mathrm{O}(\sqrt{\log n}) \quad (52)$$

as $n \to \infty$.

Proof of Lemma 4: See Appendix F. ∎

Hypothesis (b) in Theorem 1 implies that $\mathcal{V}(X \mid Y)$ is bounded away from infinity almost surely; therefore, Lemmas 2–4 yield Theorem 1, as desired. ∎

C. Proof of Theorem 2

Firstly, we establish one-shot bounds on the fundamental limits $L_{\mathrm{avg}}^*(\varepsilon, X, Y) \coloneqq L_{\mathrm{avg}}^*(1, \varepsilon, X, Y)$, namely, consider the case in which $n = 1$. The following lemma gives us a formula for $L_{\mathrm{avg}}^*(\varepsilon, X, Y)$.

Lemma 5. *Given $0 \leq \varepsilon \leq 1$, it holds that*

$$L_{\mathrm{avg}}^*(\varepsilon, X, Y) = \mathbb{E}[\langle \lfloor \log \varsigma_Y^{-1}(X) \rfloor \rangle_\varepsilon], \quad (53)$$

where ς_Y is defined in (46).

Proof of Lemma 5: See Appendix G. ∎

Unlike Lemma 1, Lemma 5 tells us that in an optimal variable-length stochastic code (F^*, G^*) achieving (53), the set of source symbols that result in an error are those with the smallest $\mathbb{P}\{X = \varsigma_Y(x)\}$. Moreover, we see that the cutoff point η of $\langle \lfloor \log \varsigma_Y^{-1}(X) \rfloor \rangle_\varepsilon$ is independent of the side-information Y, and the overflow probability of this optimal code can be evaluated as

$$\mathbb{P}\{\ell(F^*(X, Y)) \leq \eta\} \geq 1 - \varepsilon. \quad (54)$$

Using Lemma 5, we provide the following one-shot bounds on $L_{\mathrm{avg}}^*(\varepsilon, X, Y)$ in terms of the unconditional ε-cutoff entropy $\mathfrak{C}_u^\varepsilon$ defined in (26).

Lemma 6. *For every $0 \leq \varepsilon \leq 1$, it holds that*

$$\mathfrak{C}_u^\varepsilon(X \mid Y) - \log(H(X \mid Y) + 1) - \log e$$
$$\leq L_{\mathrm{avg}}^*(\varepsilon, X, Y) \leq \mathfrak{C}_u^\varepsilon(X \mid Y). \quad (55)$$

Proof of Lemma 6: Employing Proposition 1 and Lemma 5, we can prove Lemma 6 by the same manner as the proof of Lemma 2; see Appendix D for details. This completes the proof of Lemma 6. ∎

Note that Lemma 6 holds without Hypotheses (a) and (b) in Theorem 2. Since $H(X \mid Y) < \infty$ if $T_u(X \mid Y) < \infty$, and since $H(X^n \mid Y^n) = n H(X \mid Y)$ for each $n \geq 1$, Lemma 6 tells us that

$$L_{\mathrm{avg}}^*(n, \varepsilon, X, Y) = \mathfrak{C}_u^\varepsilon(X^n \mid Y^n) + \mathrm{O}(\log n) \quad (56)$$

as $n \to \infty$. Thus, it suffices to provide an appropriate asymptotic estimate on $\mathfrak{C}_u^\varepsilon(X^n \mid Y^n)$.

Lemma 7. *Given a fixed $0 \leq \varepsilon \leq 1$, it holds that*

$$\mathfrak{C}_u^\varepsilon(X^n \mid Y^n) = n(1 - \varepsilon) H(X \mid Y)$$
$$- \sqrt{n V_u(X \mid Y)} f_G(\varepsilon) + \mathrm{O}(1) \quad (57)$$

as $n \to \infty$, provided that $T_u(X \mid Y) < \infty$.

Proof of Lemma 7: Since $\iota(X^n \mid Y^n) = \iota(X_1 \mid Y_1) + \iota(X_2 \mid Y_2) + \cdots + \iota(X_n \mid Y_n)$, and since $\iota(X_1 \mid Y_1), \iota(X_2 \mid Y_2), \ldots, \iota(X_n \mid Y_n)$ are n i.i.d. real-valued r.v.'s, a naïve application of [5, Lemma 1] readily proves Lemma 7. For the readers' convenience, we now only give a sketch of the proof as follows: If $V_c(X \mid Y) = 0$, then we readily see that

$$\mathfrak{C}_u^\varepsilon(X^n \mid Y^n) = n(1 - \varepsilon) H(X \mid Y). \quad (58)$$

Thus, it suffices to consider the case where $V_c(X \mid Y) > 0$. It follows from (30) of Proposition 1 that

$$\mathfrak{C}_u^\varepsilon(X^n \mid Y^n) = n(1 - \varepsilon) H(X \mid Y)$$
$$- \int_{\eta_n}^\infty \mathbb{P}\{\iota(X^n \mid Y^n) > t\} \, \mathrm{d}t$$
$$- \varepsilon(\eta_n - n H(X \mid Y)), \quad (59)$$

where $\eta_n > 0$ is given so that

$$\mathbb{P}\{\iota(X^n \mid Y^n) > \eta_n\} + \beta_n \mathbb{P}\{\iota(X^n \mid Y^n) = \eta_n\} = \varepsilon \quad (60)$$

with an appropriate $0 \leq \beta_n < 1$. Then, the uniform Berry–Esseen bound (cf. (164) in Appendix E) shows that

$$\eta_n = n H(X \mid Y) + \sqrt{n V_u(X \mid Y)} \Phi^{-1}(1 - \varepsilon) + \mathrm{O}(1) \quad (61)$$

as $n \to \infty$, provided that $T_u(X \mid Y) < \infty$. On the other hand, it can be verified by the non-uniform Berry–Esseen bound (cf. Lemma 10 in Appendix E) that

$$\int_\eta^\infty \mathbb{P}\{\iota(X^n \mid Y^n) > t\}\, \mathrm{d}t$$
$$= \sqrt{n\, V_u(X \mid Y)} \left(f_G(\varepsilon) - \varepsilon\, \Phi^{-1}(1-\varepsilon) \right) + \mathrm{O}(1) \quad (62)$$

as $n \to \infty$, provided that $T_u(X \mid Y) < \infty$. Therefore, Lemma 7 can be proven by combining (59), (61), and (62). ∎

The proof of Theorem 2 is finally completed by combining Lemmas 6 and 7. ∎

IV. GUESSING PROBLEM

Following [13, Section III], we now introduce the guessing problem with a "giving-up" policy. Let $n \geq 1$ be an integer. Consider a stationary memoryless correlated source (X^n, Y^n) as in Section II-C. In the guessing problem, given the side-information Y^n, a guesser, equipped with a *guessing strategy* $\{x_k\}_{k=1}^\infty$ depending on Y^n, asks questions of the form "Is $X^n = x_k$?" at each time $k \in \mathbb{N}$. More precisely, a guessing strategy is induced by a *guessing function* $g_n : \mathcal{X}^n \times \mathcal{Y}^n \to \mathbb{N}$, which is a deterministic map in which $g_n(\cdot, y) : \mathcal{X}^n \to \mathbb{N}$ is bijective for each $y \in \mathcal{Y}^n$, as follows: The guesser asks "Is $g_n(X^n, Y^n) = 1$?" at time 1; if the answer is "no," the guesser asks "Is $g_n(X^n, Y^n) = 2$?" at time 2; if the answer is again "no," the guesser asks "Is $g_n(X^n, Y^n) = 3$?" at time 3, and so on. By introducing a certain error probability for guessing, the guesser can, instead of committing to a "yes/no" answer at each time $k \in \mathbb{N}$, give up at the same time k with a certain probability. For each $(k, y) \in \mathcal{X} \times \mathcal{Y}^n$, let $0 \leq \pi_n(k \mid y) \leq 1$ be a real number. The collection $\{\pi_n(k \mid y) \mid (k, y) \in \mathbb{N} \times \mathcal{Y}^n\}$ plays the role of a *giving-up policy*. Just before starting on the k-th guess, the guesser can give up his guessing task with probability $\pi_n(k \mid Y^n)$. We call the pair $(g_n, \pi_n(\cdot \mid \cdot))$ a *guessing strategy with a giving-up policy*. Formally, for each $k \geq 1$, the guesser declares an error just before starting on the k-th guess if $E_{n,1} = E_{n,2} = \cdots = E_{n,k-1} = 0$ and $E_{n,k} = 1$, where $\{E_{n,k}\}_{k=1}^\infty$ denotes a sequence of conditionally (and mutually) independent Bernoulli r.v.'s given Y^n in which

$$\mathbb{P}\{E_{n,k} = 1 \mid Y^n\} = \pi_n(k \mid Y^n) \qquad \text{(a.s.)} \qquad (63)$$

for every $k \geq 1$. Then, the *giving-up guessing function* $G_n : \mathcal{X}^n \times \mathcal{Y}^n \to \mathbb{N} \cup \{c_e\}$ is a random map given as

$$G_n(X^n, Y^n) := \begin{cases} g_n(X^n, Y^n) & \text{if } E_{n,l} = 0 \\ & \text{for all } 1 \leq l \leq g_n(X^n, Y^n), \\ c_e & \text{otherwise,} \end{cases}$$
$$(64)$$

where $c_e > 0$ denotes the cost of making an error.[6] Figure 2 illustrates a flowchart of executing a particular giving-up guessing function $G_n : \mathcal{X}^n \times \mathcal{Y}^n \to \mathbb{N} \cup \{c_e\}$.

[6]For simplicity of our analysis, we assume that c_e is not an integer. This assumption simplifies the guessing error event $\{G_n(X^n, Y^n) \neq g_n(X^n, Y^n)\}$, and does not affect the results in [13] and Corollaries 1 and 2 under a valid definition of the error event.

While Kuzuoka investigated the fundamental limits of the ρ-th moment $\mathbb{E}[G_n(X^n, Y^n)^\rho]$ with a fixed real $\rho > 0$ to evaluate the guessing cost (see [13, Equation (33)]), we are now interested in the fundamental limits of $\mathbb{E}[\log G_n(X^n, Y^n)]$. In fact, if $\mathbb{E}[G_n(X^n, Y^n)^\rho]$ is finite for some $\rho > 0$, then it follows by l'Hôpital's rule and the dominated convergence theorem that

$$\lim_{\rho \to 0^+} \frac{1}{\rho} \log \mathbb{E}[G_n(X^n, Y^n)^\rho] = \mathbb{E}[\log G_n(X^n, Y^n)], \qquad (65)$$

so our study in this section can be thought of as a limiting case of that in [13]. Noting that errors are declared if and only if $G_n(X^n, Y^n) \neq g_n(X^n, Y^n)$, we define two error formalisms as follows:

Definition 3 (Maximum error criterion). *Given a source X with side-information Y, an $(n, N, \varepsilon)_{\max}$-guessing strategy is a guessing strategy $(g_n, \pi_n(\cdot \mid \cdot))$ satisfying*

$$\mathbb{E}[\log G_n(X^n, Y^n)] \leq N, \qquad (66)$$
$$\mathbb{P}\{G_n(X^n, Y^n) \neq g_n(X^n, Y^n) \mid Y^n\} \leq \varepsilon \qquad \text{(a.s.)}. \qquad (67)$$

Definition 4 (Average error criterion). *Given a source X with side-information Y, an $(n, N, \varepsilon)_{\mathrm{avg}}$-guessing strategy is a guessing strategy $(g_n, \pi_n(\cdot \mid \cdot))$ satisfying*

$$\mathbb{E}[\log G_n(X^n, Y^n)] \leq N, \qquad (68)$$
$$\mathbb{P}\{G_n(X^n, Y^n) \neq g_n(X^n, Y^n)\} \leq \varepsilon. \qquad (69)$$

Given a probability of error $0 \leq \varepsilon \leq 1$, we investigate the two operational quantities $N_{\max}^*(n, \varepsilon, X, Y)$ and $N_{\mathrm{avg}}^*(n, \varepsilon, X, Y)$ defined in (70) and (71), respectively, and presented at the top of the next page.

Corollary 1 (Under maximum error criterion). *Suppose Hypotheses* (a) *and* (b) *in Theorem 1. Then, it holds that*

$$N_{\max}^*(n, \varepsilon, X, Y) = n\,(1-\varepsilon)\, H(X \mid Y)$$
$$- \sqrt{n\, V_c(X \mid Y)}\, f_G(\varepsilon) + \mathrm{O}(\log n) \quad (72)$$

as $n \to \infty$ for every $0 \leq \varepsilon \leq 1$.

Corollary 2 (Under average error criterion). *Suppose that $T_u(X \mid Y)$ is finite. Then, it holds that*

$$N_{\mathrm{avg}}^*(n, \varepsilon, X, Y) = n\,(1-\varepsilon)\, H(X \mid Y)$$
$$- \sqrt{n\, V_u(X \mid Y)}\, f_G(\varepsilon) + \mathrm{O}(\log n) \quad (73)$$

as $n \to \infty$ for every $0 \leq \varepsilon \leq 1$.

Proof of Corollaries 1 and 2: Relying on Theorems 1 and 2, it suffices to prove the following lemma:

Lemma 8. *For every $n \geq 1$, every $0 \leq \varepsilon \leq 1$, and every correlated source (X, Y), it holds that*

$$|N_{\max}^*(n, \varepsilon, X, Y) - L_{\max}^*(n, \varepsilon, X, Y)| \leq 1 + |\log c_e|, \qquad (74)$$
$$|N_{\mathrm{avg}}^*(n, \varepsilon, X, Y) - L_{\mathrm{avg}}^*(n, \varepsilon, X, Y)| \leq 1 + |\log c_e|. \qquad (75)$$

Lemma 8 is proven in Appendix H, completing the proof of Corollaries 1 and 2. ∎

Note that Lemma 8 is analogous to [21, Lemma 7] which is stated in terms of the moment generating functions of $\ell(F_n(X^n, Y^n))$ and $G_n(X^n, Y^n)$ in the error-free setting (i.e.,

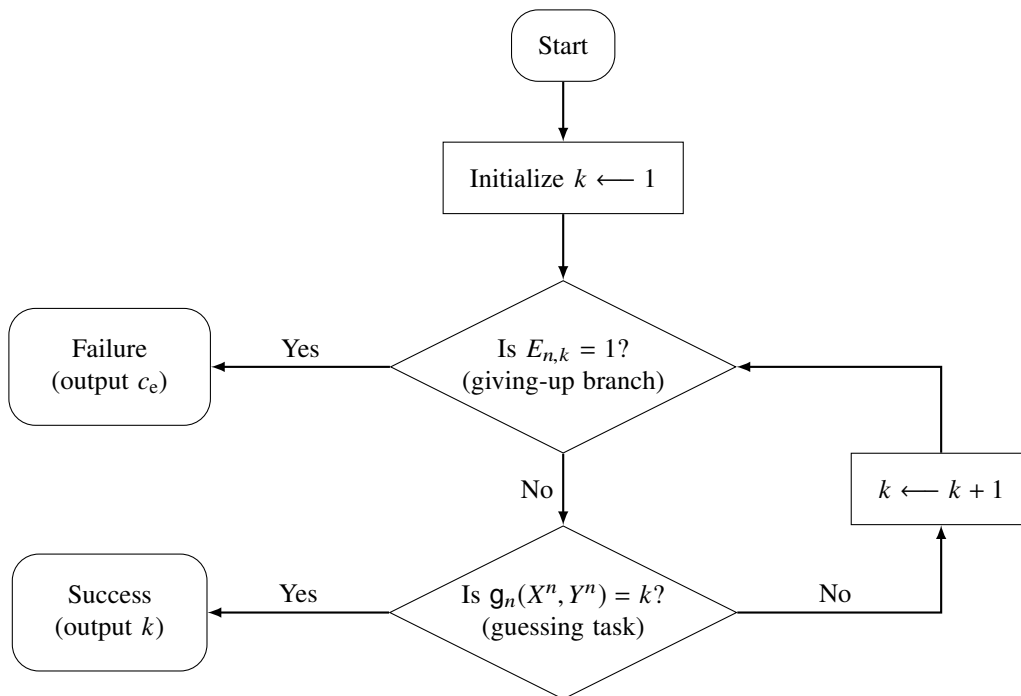Fig. 2. Flowchart of a guessing strategy with a giving-up policy (g_n, π_n).

$$N_{\max}^*(n, \varepsilon, X, Y) := \inf\{N > 0 \mid \text{there exists an } (n, N, \varepsilon)_{\max}\text{-guessing strategy for the source } X \text{ with side-information } Y\} \quad (70)$$

$$N_{\text{avg}}^*(n, \varepsilon, X, Y) := \inf\{N > 0 \mid \text{there exists an } (n, N, \varepsilon)_{\text{avg}}\text{-guessing strategy for the source } X \text{ with side-information } Y\} \quad (71)$$

$\varepsilon = 0$) and in the absence of side-information Y. From Lemma 8, it is worth pointing out that the asymptotic results of Corollaries 1 and 2 continue to hold even if the error cost c_{e} grows polynomially in n.

V. CONCLUDING REMARKS

We considered two variable-length conditional lossless source coding problems in this paper. We derived one-shot coding theorems and the second-order asymptotic results under two error formalisms: the maximum and the average probabilities of error. The one-shot bounds of Lemmas 2 and 6 are stated in terms of the ε-cutoff entropies $\mathfrak{C}_{\text{c}}^{\varepsilon}(X \mid Y)$ and $\mathfrak{C}_{\text{u}}^{\varepsilon}(X \mid Y)$, respectively. These one-shot bounds are generalizations of Kostina *et al.*'s one-shot coding theorem [5, Theorem 2] to the case in which side-information Y is available at both encoder and decoder. On the other hand, it should be noted that Kostina *et al.*'s argument in the converse proof written in [5, Section II-A] contains a minor error. Indeed, one can construct a counterexample that Kostina *et al.*'s converse proof fails to hold; see Appendix J for a counterexample. To circumvent this issue, we explicitly provided our converse proofs of Lemmas 1 and 5 in Appendices C and G, respectively. The variable-length source dispersions under the maximum and average error criteria were derived by proving asymptotic estimates on the ε-cutoff entropies $\mathfrak{C}_{\text{c}}^{\varepsilon}(X^n \mid Y^n)$ and $\mathfrak{C}_{\text{u}}^{\varepsilon}(X^n \mid Y^n)$ in Lemmas 3 and 7, respectively. These two lemmas are again generalizations of Kostina *et al.*'s asymptotic result for the

case without side-information in [5, Lemma 1] to the case in which side-information Y^n is available at both encoder and decoder. On the other hand, there is a slight subtlety in our proof, in which we have to establish the *uniform boundedness property* of the remainder term, represented by the quantity $+O(1)$, with respect to the side-information Y^n; see Lemma 9 stated in Appendix E for a technical statement of this uniform boundedness property. In Section IV, we showed that our results can be applicable to Kuzuoka's guessing problem [13, Section III].

Recently, the present authors [31], [32] derived asymptotic expansions of smooth Rényi entropies and their conditional versions and using these expansions, established fundamental limits of various information-theoretic problems including Campbell's source coding problem [19], the guessing problem studied by Massey [17] and Arıkan [18] (see also recent work by Kuzuoka [13]), and the task encoding problem [33], all allowing errors. We then showed that the *first-order terms* of the asymptotic expansions of these fundamental limits differ under average and maximum error criteria, where the the average and maximum are taken with respect to the available side information Y, as in Definitions 1 and 2. The differences between the asymptotic expansions derived in [31], [32] and the main results in this paper arise from the following differences in the problem settings: While Theorems 1 and 2 evaluate the ordinary expectation $\mathbb{E}[\ell(F_n(X^n, Y^n))]$, Campbell's source coding problem considered in [31], [32] evaluates the cumulant gener-

ating function of codeword lengths, i.e., $\log \mathbb{E}[2^{\rho \ell(F_n(X^n, Y^n))}]$ for fixed $\rho > 0$. Similarly, while Corollaries 1 and 2 evaluate the right-hand side of (65), the guessing problem considered in [31], [32] evaluates the left-hand side of (65) without taking the limit as $\rho \to 0^+$. Moreover, Campbell's source coding problem imposes a prefix-free constraint. Finally, it is worth mentioning that the difference between the first-order terms derived in [31], [32] under the average and maximum error criteria can be characterized by the law of total variance, as in (20), Theorems 1 and 2, and Corollaries 1 and 2.

In this study, we investigated the infimum of $L > 0$ in which $\mathbb{E}[\ell(F_n(X^n, Y^n))] \leq L$ under the average error criterion $\mathbb{P}\{X \neq G_n(F_n(X^n, Y^n), Y^n)\} \leq \varepsilon$ and the maximum error criterion $\mathbb{P}\{X \neq G_n(F_n(X^n, Y^n), Y^n) \mid Y^n\} \leq \varepsilon$ (a.s.); see Definitions 1 and 2, respectively. A natural avenue for future research is to examine the infimum of $L > 0$ in which $\mathbb{E}[\ell(F_n(X^n, Y^n)) \mid Y^n] \leq L$ (a.s.) under these error criteria. In addition, comparing the present setting and results to the variable-length Slepian–Wolf coding problem [16], [27]–[29] would be of interest.

APPENDIX A
PROOF OF PROPOSITION 1

A. Proofs of (28) and (29)

The identity (28) is stated in [5, Equation (38)], and can be thought of as a special case of (29) in which $\sigma(W)$ is the trivial σ-algebra $\{\emptyset, \Omega\}$. Thus, it suffices to prove (29).

Let $\epsilon^* : [0, \infty) \times \mathcal{W} \to [0, 1]$ be the measurable map given as

$$\epsilon^*(z, W) = \begin{cases} 0 & \text{if } z < \eta_W, \\ \beta_W & \text{if } z = \eta_W, \\ 1 & \text{if } z > \eta_W \end{cases} \quad (76)$$

for each $z \geq 0$, where the $\sigma(W)$-measurable r.v.'s $\eta_W \geq 0$ and $0 \leq \beta_W < 1$ are given in (25). It is clear from (25) and (76) that

$$\mathbb{E}[\epsilon^*(Z, W) \mid W] = \varepsilon \qquad \text{(a.s.)}. \quad (77)$$

After some algebra, we get

$$\mathbb{P}\{\langle Z \mid W \rangle_\varepsilon > t \mid W\} = \mathbb{P}\{(1 - \epsilon^*(Z, W)) Z > t \mid W\}$$
$$= \begin{cases} \mathbb{P}\{Z > t \mid W\} & \text{if } t < 0, \\ \mathbb{P}\{Z > t \mid W\} - \varepsilon & \text{if } 0 \leq t < \eta_W, \\ 0 & \text{if } t \geq \eta_W \end{cases}$$
$$\quad (78)$$

a.s. Therefore, the two r.v.'s $\langle Z \mid W \rangle_\varepsilon$ and $(1 - \epsilon^*(Z, W)) Z$ are equal in distribution, which implies that

$$\mathbb{E}[\langle Z \mid W \rangle_\varepsilon] = \mathbb{E}[(1 - \epsilon^*(Z, W)) Z]. \quad (79)$$

Consider an arbitrary measurable map $\epsilon : [0, \infty) \times \mathcal{W} \to [0, 1]$ satisfying

$$\mathbb{E}[\epsilon(Z, W) \mid W] \leq \varepsilon \qquad \text{(a.s.)}. \quad (80)$$

Denoting by $\mathbf{1}_A$ the indicator function of $A \subset \Omega$, a direct calculation shows that (81) written in the top of the next page holds, where

- (a) and (c) follow from the definition of ϵ^* stated in (76),
- (b) follows from the fact that η_W is $\sigma(W)$-measurable, and
- (d) follows from (77) and (80) and the fact that $\eta_W \geq 0$.

Combining (77) and (79)–(81), we obtain (29) of Proposition 1, as desired. ∎

B. Proofs of (30) and (31)

The identity (30) can be shown in the same manner as [5, Equations (155)–(156)], and can be thought of as a special case of (31) in which $\sigma(W)$ is the trivial σ-algebra. Thus, it suffices to prove (31).

It can be verified that the following conditional version of [5, Equation (157)] holds:[7]

$$\mathbb{E}[Z \mathbf{1}_{\{Z > z\}} \mid W] = \int_z^\infty \mathbb{P}\{Z > t \mid W\} \, dt$$
$$+ z \mathbb{P}\{Z > z \mid W\} \quad \text{(a.s.)} \quad (82)$$

for every real number $z \geq 0$.

We have

$$\mathbb{E}[\langle Z \mid W \rangle_\varepsilon \mid W]$$
$$\overset{(a)}{=} \mathbb{E}[Z \mathbf{1}_{\{Z < \eta_W\}} \mid W] + \eta_W (1 - \beta_W) \mathbb{P}\{Z = \eta_W \mid W\}$$
$$= \mathbb{E}[Z \mid W] - \mathbb{E}[Z \mathbf{1}_{\{Z > \eta_W\}} \mid W] - \eta_W \beta_W \mathbb{P}\{Z = \eta_W \mid W\}$$
$$\overset{(b)}{=} \mathbb{E}[Z \mid W] - \int_{\eta_W}^\infty \mathbb{P}\{Z > t \mid W\} \, dt$$
$$- \eta_W \left(\mathbb{P}\{Z > \eta_W \mid W\} + \beta_W \mathbb{P}\{Z = \eta_W \mid W\} \right)$$
$$\overset{(c)}{=} \mathbb{E}[Z \mid W] - \int_{\eta_W}^\infty \mathbb{P}\{Z > t \mid W\} \, dt - \varepsilon \eta_W \quad \text{(a.s.)}, \quad (83)$$

where

- (a) follows from the definition of $\langle \cdot \mid \cdot \rangle_\varepsilon$ stated in (23),
- (b) follows from (82) by noting that η_W is $\sigma(W)$-measurable, and
- (c) follows from (25).

This completes the proof of (31) of Proposition 1. ∎

C. Proof of (32)

Since the functional $\epsilon \mapsto \mathbb{E}[(1 - \epsilon(Z)) Z]$ of a mapping $\epsilon : [0, \infty) \to [0, 1]$ is linear, we readily see that

$$\min_{\epsilon : \mathbb{E}[\epsilon(Z)] = \varepsilon} \mathbb{E}[(1 - \epsilon(Z)) Z]$$
$$= \min_{\epsilon : \mathbb{E}[\epsilon(Z, W)] = \varepsilon} \mathbb{E}[(1 - E(Z)) Z], \quad (84)$$

where the minimization in the left-hand side (resp. the right-hand side) is taken over the maps $\epsilon : [0, \infty) \to [0, 1]$ (resp. the maps $\epsilon : [0, \infty) \times \mathcal{W} \to [0, 1]$) satisfying $\mathbb{E}[\epsilon(Z)] = \varepsilon$ (resp. $\mathbb{E}[\epsilon(Z, W)] = \varepsilon$). Therefore, the proof is completed by the identities (28) and (29) of Proposition 1, and the fact that $\mathbb{E}[\epsilon(Z, W)] = \varepsilon$ if $\mathbb{E}[\epsilon(Z, W) \mid W] = \varepsilon$ a.s. ∎

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[7] While [5, Equation (157)] can be verified by applying Tonelli's theorem only once, one can show (82) by applying Tonelli's theorem twice.

$$\mathbb{E}[(\epsilon(Z,W) - \epsilon^*(Z,W))\, Z \mid W]$$

$$= \mathbb{E}[(\epsilon(Z,W) - \epsilon^*(Z,W))\, Z\, (\mathbf{1}_{\{Z < \eta_W\}} + \mathbf{1}_{\{Z = \eta_W\}} + \mathbf{1}_{\{Z > \eta_W\}}) \mid W]$$

$$\overset{(a)}{=} \mathbb{E}[\epsilon(Z,W)\, Z\, \mathbf{1}_{\{Z < \eta_W\}} \mid W] + \mathbb{E}[(\epsilon(Z,W) - \beta_W)\, Z\, \mathbf{1}_{\{Z = \eta_W\}} \mid W] + \mathbb{E}[(\epsilon(Z,W) - 1)\, Z\, \mathbf{1}_{\{Z > \eta_W\}} \mid W]$$

$$\overset{(b)}{\leq} \eta_W \left(\mathbb{E}[\epsilon(Z,W)\, \mathbf{1}_{\{Z < \eta_W\}} \mid W] + \mathbb{E}[(\epsilon(Z,W) - \beta_W)\, \mathbf{1}_{\{Z = \eta_W\}} \mid W] + \mathbb{E}[(\epsilon(Z,W) - 1)\, \mathbf{1}_{\{Z > \eta_W\}} \mid W] \right)$$

$$\overset{(c)}{=} \eta_W\, \mathbb{E}[\epsilon(Z,W) - \epsilon^*(Z,W) \mid W]$$

$$\overset{(d)}{\leq} 0 \qquad \text{(a.s.)}, \tag{81}$$

D. Proofs of (33) and (34)

The identity (33) is a special case of (34) in which $\sigma(W)$ is the trivial σ-algebra. Hence, it suffices to prove (34). Choose two $\sigma(W)$-measurable real-valued r.v.'s $\eta_W^{(1)}$ and $\eta_W^{(2)}$ so that

$$\mathbb{P}\{Z_1 > \eta_W^{(1)} \mid W\} + \beta_W^{(1)}\, \mathbb{P}\{Z_1 = \eta_W^{(1)} \mid W\} = \varepsilon \quad \text{(a.s.)}, \tag{85}$$

$$\mathbb{P}\{Z_2 > \eta_W^{(2)} \mid W\} + \beta_W^{(2)}\, \mathbb{P}\{Z_2 = \eta_W^{(2)} \mid W\} = \varepsilon \quad \text{(a.s.)} \tag{86}$$

for some $\sigma(W)$-measurable real-valued r.v.'s $\beta_W^{(1)}$ and $\beta_W^{(2)}$. Then, it follows by the premise of (34) that

$$\eta_W^{(1)} \geq \eta_W^{(2)} \qquad \text{(a.s.)}. \tag{87}$$

Now, a direct calculation shows

$$\mathbb{E}[\langle Z_1 \mid W \rangle_\varepsilon \mid W]$$

$$\overset{(a)}{=} \int_0^{\eta_W^{(1)}} \mathbb{P}\{Z_1 > t \mid W\}\, \mathrm{d}t - \varepsilon\, \eta_W^{(1)}$$

$$\overset{(b)}{=} \int_0^{\eta_W^{(2)}} \mathbb{P}\{Z_1 > t \mid W\}\, \mathrm{d}t + \int_{\eta_W^{(1)}}^{\eta_W^{(2)}} \mathbb{P}\{Z_1 > t \mid W\}\, \mathrm{d}t - \varepsilon\, \eta_W^{(1)}$$

$$\overset{(c)}{\leq} \int_0^{\eta_W^{(2)}} \mathbb{P}\{Z_1 > t \mid W\}\, \mathrm{d}t + \int_{\eta_W^{(1)}}^{\eta_W^{(2)}} \varepsilon\, \mathrm{d}t - \varepsilon\, \eta_W^{(1)}$$

$$= \int_0^{\eta_W^{(2)}} \mathbb{P}\{Z_1 > t \mid W\}\, \mathrm{d}t - \varepsilon\, \eta_W^{(2)}$$

$$\overset{(d)}{\leq} \int_0^{\eta_W^{(2)}} \mathbb{P}\{Z_2 > t \mid W\}\, \mathrm{d}t - \varepsilon\, \eta_W^{(2)}$$

$$\overset{(e)}{=} \mathbb{E}[\langle Z_2 \mid W \rangle_\varepsilon \mid W] \tag{88}$$

a.s., where

- (a) follows from (31),
- (b) follows from (87),
- (c) follows from the fact that $\mathbb{P}\{Z_1 > t \mid W\} \leq \varepsilon$ if $t \geq \eta_W^{(1)}$ a.s. (see (85)),
- (d) follows by the premise of (34), and
- (e) follows again from (31).

Therefore, the conclusion of (34) holds, completing the proof. ∎

Let $\mathcal{X} = \mathcal{Y} = \{1, 2, \dots\}$. Suppose that

$$P_Y(y) = \frac{6}{\pi^2\, y^2}, \tag{89}$$

$$P_{X|Y}(x \mid y) = \left(1 - \frac{1}{y}\right)^{x-1} \frac{1}{y} \tag{90}$$

for each $(x, y) \in \mathcal{X} \times \mathcal{Y}$. Namely, the conditional distribution $P_{X|Y}(\cdot \mid y)$ is the geometric distribution with parameter $1/y$ for each $y \in \mathcal{Y}$. After some algebra, we have[8]

$$\mathcal{V}(X \mid y) = y\, (y - 1) \left(\log \frac{y}{y - 1} \right)^2 \tag{91}$$

for each $y \in \mathcal{Y}$, implying that Hypotheses (a) and (b) in Theorem 1 fail to holds. Therefore, Theorem 1 cannot ensure that (42) holds for this source (X, Y).

On the other hand, it can be verified that

$$H(X \mid Y) \leq 2 + \frac{6}{\pi^2} \sum_{y=2}^{\infty} \frac{\log(y - 1)}{y^2}, \tag{92}$$

$$\tilde{\mathcal{T}}(X \mid y) := \sum_{x \in \mathcal{X}} P_{X|Y}(x \mid y) \left(\log \frac{1}{P_{X|Y}(x \mid y)} \right)^3$$

$$\leq \log^3(y - 1) + 6 \log^2(y - 1)$$
$$\quad + 12 \log(y - 1) + 12 \tag{93}$$

for each $y \geq 2$, where $\tilde{\mathcal{T}}(X \mid 1) = 0$. Thus, we observe that

$$T_{\mathrm{u}}(X \mid Y) \leq \mathbb{E}[\tilde{\mathcal{T}}(X \mid Y)] + H(X \mid Y)^3$$

$$\leq \frac{6}{\pi^2} \sum_{y=2}^{\infty} \frac{\log^3(y-1) + 6\log^2(y-1) + 12\log(y-1)}{y^2}$$

$$\quad + 12 + \left(2 + \frac{6}{\pi^2} \sum_{y=2}^{\infty} \frac{\log(y-1)}{y^2} \right)^3, \tag{94}$$

implying that $T_{\mathrm{u}}(X \mid Y)$ is finite. Therefore, it follows from Theorem 2 that (43) holds for this source (X, Y).

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[8]We adopt the usual convention that $0 \log 0 = 0$ and $0 \log^s(1/0) = 0$ for $s \geq 1$.

APPENDIX C
PROOF OF LEMMA 1

Throughout Appendix C, we consider one-shot $(n = 1)$ variable-length stochastic codes (F, G) as defined in Section II-C. We say that a decoder G is *deterministic* if $G(F(X, Y), X)$ is $\sigma(F(X, Y), X)$-measurable. To specify the determinism, we use the lower case g to denote a deterministic decoder.

Consider an $(L, \varepsilon)_{\max}$-code (F, G) satisfying

$$\mathbb{E}[\ell(F(X, Y))] \le L, \tag{95}$$

$$\mathbb{P}\{X \ne G(F(X, Y), Y) \mid Y\} \le \varepsilon \quad \text{(a.s.)}. \tag{96}$$

It can be verified that there exists a deterministic decoder g_0 satisfying[9]

$$\mathbb{P}\{X \ne g_0(F(X, Y), Y) \mid Y\} \le \mathbb{P}\{X \ne G(F(X, Y), Y) \mid Y\} \tag{97}$$

a.s. In addition, for each $(x, y) \in \mathcal{X} \times \mathcal{Y}$, construct another stochastic encoder F_0 as

$$F_0(x, y) := \begin{cases} \varnothing & \text{if } x \ne g_0(F(x, y), y), \\ F(x, y) & \text{otherwise.} \end{cases} \tag{98}$$

As shown later, the new code (F_0, g_0) has a better performance than that of the initial code (F, G). It is clear that

$$F_0(X, Y) \ne \varnothing \implies X = g_0(F_0(X, Y), Y). \tag{99}$$

Now, generate a random collection $\mathcal{B}(Y)$ of subsets of $\{0, 1\}^*$ as

$$\mathcal{B}(Y) := \{\mathcal{B}(x \mid Y) \mid x \in \mathcal{X}\} \setminus \{\emptyset\}, \tag{100}$$

where the random subset $\mathcal{B}(x \mid Y)$ of $\{0, 1\}^*$ is defined by

$$\mathcal{B}(x \mid Y)$$
$$:= \{\boldsymbol{b} \in \{0, 1\}^* \setminus \{\varnothing\} \mid \mathbb{P}\{F_0(x, Y) = \boldsymbol{b} \mid Y\} > 0\} \tag{101}$$

if $x \ne g_0(\varnothing, Y)$; and $\mathcal{B}(x \mid Y) := \{\varnothing\}$ otherwise. We shall prove the disjointness of the sets $\mathcal{B}(x \mid Y)$, $x \in \mathcal{X}$, in $\mathcal{B}(Y)$ as follows: Choose $\boldsymbol{b}' \in \{0, 1\}^* \setminus \{\varnothing\}$ and $x_1, x_2 \in \mathcal{X}$ so that $\mathbb{P}(\mathcal{E}_1 \cap \mathcal{E}_2) > 0$, where the two events $\mathcal{E}_1$ and $\mathcal{E}_2$ are given by

$$\mathcal{E}_1 := \{\mathbb{P}\{F_0(x_1, Y) = \boldsymbol{b}' \mid Y\} > 0\}, \tag{102}$$
$$\mathcal{E}_2 := \{\mathbb{P}\{F_0(x_2, Y) = \boldsymbol{b}' \mid Y\} > 0\}, \tag{103}$$

respectively. It is clear that $\mathcal{E}_i \in \sigma(Y)$ for each $i = 1, 2$. Moreover, it follows from (99) that $g_0(\boldsymbol{b}', Y) = x_i$ on the event $\mathcal{E}_i$ for each $i = 1, 2$. Thus, since $g_0(\boldsymbol{b}', Y)$ is $\sigma(Y)$-measurable, we observe that $x_1 = x_2$ whenever $\mathbb{P}(\mathcal{E}_1 \cap \mathcal{E}_2) > 0$. Therefore, the random collection $\mathcal{B}(Y)$ is disjoint a.s., i.e.,

$$\mathbb{P}\{\mathcal{B}(x_1, Y) \cap \mathcal{B}(x_2, Y) = \emptyset \text{ for all } x_1 \ne x_2\} = 1. \tag{104}$$

By the disjointness of (104), one can find an index set $\mathcal{I}(Y) = \{1, 2, \ldots, |\mathcal{B}(Y)|\}$ of the collection $\mathcal{B}(Y)$ so that

$$\mathcal{B}(Y) = \{\mathcal{B}_i(Y) \mid i \in \mathcal{I}(Y)\} \tag{105}$$

[9]Note that in general, the determinism of decoders is not a necessary condition to be optimal.

and

$$\mathbb{P}\left\{\begin{matrix} \text{for all } i < j, \text{ there exists } \boldsymbol{b} \in \mathcal{B}_i(Y) \text{ s.t.} \\ \boldsymbol{b} \prec \tilde{\boldsymbol{b}} \text{ for all } \tilde{\boldsymbol{b}} \in \mathcal{B}_j(Y) \end{matrix}\right\} = 1, \tag{106}$$

where the binary relation $\prec$ on $\{0, 1\}^*$ represents the lexicographical order in $\{0, 1\}^*$. Let $\{\boldsymbol{b}_i\}_{i=1}^{\infty}$ be the lexicographical ordering of the strings in $\{0, 1\}^*$ so that $\boldsymbol{b}_i \prec \boldsymbol{b}_j$ whenever $i < j$; e.g., $\boldsymbol{b}_1 = \varnothing$, $\boldsymbol{b}_2 = 0$, $\boldsymbol{b}_3 = 1$, $\boldsymbol{b}_4 = 00$, $\boldsymbol{b}_5 = 01$, $\boldsymbol{b}_6 = 10$, $\boldsymbol{b}_7 = 11$, $\boldsymbol{b}_8 = 000$, and so on. It is trivial that

$$\mathbb{P}\{\mathcal{B}_1(Y) = \{\varnothing\}\} = 1; \tag{107}$$

consequently, it follows from (104) that

$$\mathbb{P}\{\varnothing \notin \mathcal{B}_i(Y) \text{ for all } i \in \mathcal{I}(Y) \setminus \{1\}\} = 1. \tag{108}$$

Now, define the event

$$\mathcal{A}_k$$
$$:= \left\{\begin{matrix} \ell(\boldsymbol{b}_i) \le \ell(\boldsymbol{b}) \text{ for all } 1 \le i \le \min\{k, |\mathcal{B}(Y)|\} \\ \text{and } \boldsymbol{b} \in \mathcal{B}_i(Y) \end{matrix}\right\} \tag{109}$$

for each integer $k \ge 1$. It can be verified from (106) and (107) by induction that

$$\mathbb{P}(\mathcal{A}_k) = 1 \tag{110}$$

for every $k \ge 1$. Hence, it follows from the monotonicity $\mathcal{A}_1 \supset \mathcal{A}_2 \supset \mathcal{A}_3 \supset \cdots$ that

$$\mathbb{P}\{\ell(\boldsymbol{b}_i) \le \ell(\boldsymbol{b}) \text{ for all } i \in \mathcal{I}(Y) \text{ and } \boldsymbol{b} \in \mathcal{B}_i(Y)\} = 1. \tag{111}$$

Based on the previous paragraph, define the random map $\Phi_Y : \{0, 1\}^* \to \{0, 1\}^*$ so that

$$\Phi_Y(\boldsymbol{b}) := \begin{cases} \varnothing & \text{if } \boldsymbol{b} \notin \mathcal{B}_i(Y) \text{ for all } i \in \mathcal{I}(Y), \\ \boldsymbol{b}_i & \text{if } \boldsymbol{b} \in \mathcal{B}_i(Y) \text{ for some } i \in \mathcal{I}(Y). \end{cases} \tag{112}$$

Moreover, the disjointness of (104) ensures the existence of a random map $\Psi_Y : \mathcal{X} \to \mathcal{X} \cup \{0\}$ satisfying

$$\Psi_Y(i) = \begin{cases} x & \text{if } \mathcal{B}_i(Y) = \mathcal{B}(x \mid Y) \text{ for some } x \in \mathcal{X}, \\ 0 & \text{otherwise.} \end{cases} \tag{113}$$

Note that $\Phi_Y(\boldsymbol{b})$ and $\Psi_Y(i)$ are $\sigma(Y)$-measurable for each $\boldsymbol{b} \in \{0, 1\}^*$ and $i \in \mathcal{X}$, respectively. Then, construct another variable-length stochastic code (F_1, g_1) so that

$$F_1(x, Y) := \Phi_Y(F_0(x, Y)), \tag{114}$$
$$g_1(\boldsymbol{b}, Y) := \Psi_Y(i) \quad \text{if } \boldsymbol{b} = \boldsymbol{b}_i \text{ for some } i \ge 1. \tag{115}$$

Now, we shall evaluate the average codeword length of the encoder F_1. A direct calculation shows

$$\mathbb{E}[\ell(F_1(X, Y))]$$
$$\overset{(a)}{=} \mathbb{E}\left[\sum_{i=2}^{|\mathcal{B}(Y)|} \ell(\boldsymbol{b}_i) \mathbf{1}_{\{F_0(X,Y) \in \mathcal{B}_i(Y)\}}\right]$$
$$\overset{(b)}{\le} \mathbb{E}\left[\sum_{i=2}^{|\mathcal{B}(Y)|} \ell(F_0(X, Y)) \mathbf{1}_{\{F_0(X,Y) \in \mathcal{B}_i(Y)\}}\right]$$
$$\overset{(c)}{\le} \mathbb{E}\left[\ell(F_0(X, Y)) \mathbf{1}_{\{X \ne g_0(\varnothing, Y)\}}\right]$$
$$\overset{(d)}{=} \mathbb{E}\left[\ell(F(X, Y)) \mathbf{1}_{\{X = g_0(F(X,Y), Y)\} \cap \{X \ne g_0(\varnothing, Y)\}}\right]$$
$$\le \mathbb{E}[\ell(F(X, Y))]$$
$$\overset{(e)}{\le} L, \tag{116}$$

where

- (a) follows from the definitions of $\mathcal{B}(Y)$ and ϕ stated in (105) and (112), respectively, and the fact that $\ell(\boldsymbol{b}_1 = \varnothing) = 0$,
- (b) follows from (111),
- (c) follows from the disjointness of $\mathcal{B}(Y)$ stated in (104) and the fact that

$$\mathbb{P}\{F_0(X,Y) \in \mathcal{B}_i(Y) \text{ for some } 1 \le i \le |\mathcal{B}(Y)|\} = 1; \quad (117)$$

that is,

$$\sum_{i=2}^{|\mathcal{B}(Y)|} \mathbf{1}_{\{F_0(X,Y)\in\mathcal{B}_i(Y)\}}$$
$$= \mathbf{1}_{\{F_0(X,Y)\in\mathcal{B}_i(Y) \text{ for some } 2\le i\le|\mathcal{B}(Y)|\}}$$
$$= \mathbf{1}_{\{F_0(X,Y)\ne\varnothing\}}$$
$$= \mathbf{1}_{\{X\ne g_0(\varnothing,Y)\}\cap\{X=g_0(F_0(X,Y),Y)\}}$$
$$\le \mathbf{1}_{\{X\ne g_0(\varnothing,Y)\}} \quad (118)$$

a.s.,

- (d) follows by the definition of F_0 stated in (98), and
- (e) follows from (95).

Namely, the average codeword length of the encoder F_1 is shorter than or equal to that of the initial encoder F.

Next, we shall evaluate the error probability of the code (F_1, g_1). We observe that

$$\mathbb{P}\{X \ne g_1(F_1(X,Y),Y) \mid Y\}$$
$$= \sum_{i=1}^{\infty} \mathbb{P}\{F_1(X,Y) = \boldsymbol{b}_i \text{ and } X \ne g_1(\boldsymbol{b}_i,Y) \mid Y\}$$
$$\overset{(a)}{=} \sum_{i=1}^{|\mathcal{B}(Y)|} \mathbb{P}\{\Phi_Y(F_0(X,Y)) = \boldsymbol{b}_i \text{ and } X \ne \Psi_Y(i) \mid Y\}$$
$$\overset{(b)}{=} \sum_{i=1}^{|\mathcal{B}(Y)|} \mathbb{P}\{F_0(X,Y) \in \mathcal{B}_i(Y) \text{ and } X \ne \Psi_Y(i) \mid Y\}$$
$$\overset{(c)}{=} \sum_{i=1}^{|\mathcal{B}(Y)|} \mathbb{P}\{F_0(X,Y) \in \mathcal{B}_i(Y) \text{ and } \mathcal{B}(X,Y) \ne \mathcal{B}_i(Y) \mid Y\}$$
$$\overset{(d)}{=} \mathbb{P}\{F_0(X,Y) \in \mathcal{B}_1(Y) \text{ and } \mathcal{B}(X,Y) \ne \mathcal{B}_1(Y) \mid Y\}$$
$$\overset{(e)}{=} \mathbb{P}\{F_0(X,Y) = \varnothing \text{ and } X \ne g_0(\varnothing,Y) \mid Y\}$$
$$\overset{(f)}{=} \mathbb{P}\{X \ne g_0(F_0(X,Y),Y) \text{ and } X \ne g_0(\varnothing,Y) \mid Y\}$$
$$\overset{(g)}{=} \mathbb{P}\{X \ne g_0(F(X,Y),Y) \text{ and } X \ne g_0(\varnothing,Y) \mid Y\}$$
$$\le \mathbb{P}\{X \ne g_0(F(X,Y),Y) \mid Y\}$$
$$\overset{(h)}{\le} \varepsilon \quad \text{(a.s.),} \quad (119)$$

where

- (a) follows by the definition of (F_1, g_1) stated in (114) and (115),
- (b) follows by the definition of the random map $\Phi_Y(\cdot)$ stated in (112),
- (c) follows by the definition of the random map $\Psi_Y(\cdot)$ stated in (113),
- (d) follows from the fact that $F_0(X,Y) \in \mathcal{B}_i(Y)$ only if $\mathcal{B}(X,Y) = \mathcal{B}_i(Y)$ for each $2 \le i \le |\mathcal{B}(Y)|$ a.s.,

- (e) follows from (107),
- (f) follows from the fact that $F_0(X,Y) = \varnothing$ if and only if $X = g(\varnothing,Y)$ or $X \ne g_0(F_0(X,Y),Y)$,
- (g) follows by the definition of F_0 stated in (98), and
- (h) follows from (96) and (97).

In other words, the maximum probability of error for the code (F_1, g_1) is smaller than or equal to that for the initial code (F, G).

Here, we shall prove the converse result of Lemma 1, i.e., we shall show that the average codeword length of (F, G) satisfying (96) is always bounded from below by the right-hand side of (45). We see that

$$1 - \varepsilon \overset{(a)}{\le} \mathbb{P}\{X = g_1(F_1(X,Y),Y) \mid Y\}$$
$$= \mathbb{P}\{X = g_1(\varnothing,Y) \text{ and } F_1(X,Y) = \varnothing \mid Y\}$$
$$+ \mathbb{P}\{X = g_1(F_1(X,Y),Y) \text{ and } F_1(X,Y) \ne \varnothing \mid Y\}$$
$$\overset{(b)}{=} \mathbb{P}\{X = \Psi_Y(1) \mid Y\}$$
$$+ \mathbb{P}\{X = g_1(F_1(X,Y),Y) \text{ and } F_1(X,Y) \ne \varnothing \mid Y\}$$
$$= \mathbb{P}\{X = \Psi_Y(1) \mid Y\}$$
$$+ \mathbb{P}\left\{ \begin{array}{l} X = g_1(\boldsymbol{b}_i,Y) \text{ and} \\ F_1(X,Y) = \boldsymbol{b}_i \text{ for some } i \ge 2 \end{array} \middle| Y \right\}$$
$$\overset{(c)}{=} \mathbb{P}\{X = \Psi_Y(1) \mid Y\}$$
$$+ \mathbb{P}\{X = \Psi_Y(i) \text{ for some } 2 \le i \le |\mathcal{B}(Y)| \mid Y\}$$
$$\overset{(d)}{=} \sum_{i=1}^{|\mathcal{B}(Y)|} P_{X|Y}(\Psi_Y(i) \mid Y) \quad \text{(a.s.),} \quad (120)$$

where

- (a) follows from (119),
- (b) and (c) follows from the definition of (F_1, g_1) stated in (114) and (115), and
- (d) follows from the fact that $\Psi_Y(i) \ne \Psi_Y(j)$ if $1 \le i < j \le |\mathcal{B}(Y)|$ a.s.

Now, define the $\sigma(Y)$-measurable r.v.'s ξ_Y and ζ_Y so that[10]

$$\xi_Y := \sup\left\{ k \ge 0 \,\middle|\, \sum_{i=1}^{k} P_{X|Y}(\Psi_Y(i) \mid Y) \le 1 - \varepsilon \right\}, \quad (121)$$

$$\zeta_Y := 1 - \varepsilon - \sum_{i=1}^{\xi_Y} P_{X|Y}(\Psi_Y(i) \mid Y), \quad (122)$$

respectively. In addition, define the $\sigma(Y)$-measurable r.v.'s κ_Y and γ_Y so that

$$\kappa_Y := \sup\left\{ k \ge 0 \,\middle|\, \sum_{x=1}^{k} P_{X|Y}(\varsigma_Y(x) \mid Y) \le 1 - \varepsilon \right\}, \quad (123)$$

$$\gamma_Y := 1 - \varepsilon - \sum_{x=1}^{\gamma_Y} P_{X|Y}(\varsigma_Y(x) \mid Y), \quad (124)$$

---

[10]Note that $\xi_Y = 0$ if $P_{X|Y}(\psi(1,Y) \mid Y) \ge 1 - \varepsilon$; and $\xi_Y = \infty$ if $\sum_{i=1}^{\infty} P_{X|Y}(\psi(i,Y) \mid Y) = 1 - \varepsilon$ and $P_{X|Y}(\psi(i,Y) \mid Y) > 0$ for all $i \ge 1$.

respectively, where ς_Y is given in (46). Furthermore, define

$$p_1(k \mid Y) := \begin{cases} P_{X|Y}(\Psi_Y(k) \mid Y) & \text{if } 1 \le k \le \xi_Y, \\ \zeta_Y & \text{if } k = \xi_Y + 1, \\ 0 & \text{if } \xi_Y + 2 \le k < \infty. \end{cases} \quad (125)$$

$$p_2(x \mid Y) := \begin{cases} P_{X|Y}(\varsigma_Y(x) \mid Y) & \text{if } 1 \le x \le \kappa_Y, \\ \gamma_Y & \text{if } x = \kappa_Y + 1, \\ 0 & \text{if } \kappa_Y + 2 \le x < \infty. \end{cases} \quad (126)$$

Then, a direct calculation shows

$$\mathbb{E}[\ell(F_1(X,Y)) \mid Y] \ge \sum_{k=1}^{\infty} \lfloor \log k \rfloor \, p_1(k \mid Y)$$

$$= \sum_{j=0}^{\infty} j \sum_{k=2^j}^{2^{j+1}-1} p_1(k \mid Y)$$

$$= \sum_{j=1}^{\infty} \sum_{k=2^j}^{\infty} p_1(k \mid Y), \quad (127)$$

$$\mathbb{E}[\langle \lfloor \log \varsigma_Y^{-1}(X) \rfloor \mid Y \rangle_\varepsilon \mid Y] = \sum_{x=1}^{\infty} \lfloor \log x \rfloor \, p_2(x \mid Y)$$

$$= \sum_{j=0}^{\infty} j \sum_{x=2^j}^{2^{j+1}-1} p_2(x \mid Y)$$

$$= \sum_{j=1}^{\infty} \sum_{x=2^j}^{\infty} p_2(x \mid Y) \quad (128)$$

a.s., respectively, where the second equalities in (127) and (128) follow from the fact that $\lfloor \log k \rfloor = j$ if and only if $2^j \le k < 2^{j+1}$ for every $k \ge 1$. On the other hand, since ς_Y rearranges the probability masses in $P_{X|Y}(\cdot \mid Y)$ in non-increasing order (see (46)), it can be verified that $p_1(\cdot \mid Y)$ is majorized by $p_2(\cdot \mid Y)$ a.s., i.e., it follows that

$$\sum_{k=1}^{l} p_1(k \mid Y) \le \sum_{x=1}^{l} p_2(x \mid Y) \quad \text{(a.s.)} \quad (129)$$

for every $l \ge 1$, and

$$\sum_{k=1}^{\infty} p_1(k \mid Y) = \sum_{x=1}^{\infty} p_2(x \mid Y) = 1 - \varepsilon \quad \text{(a.s.).} \quad (130)$$

Therefore, it follows from (127)–(130) that

$$\mathbb{E}[\ell(F_1(X,Y))] \ge \mathbb{E}[\langle \lfloor \log \varsigma_Y^{-1}(X) \rfloor \mid Y \rangle_\varepsilon]. \quad (131)$$

Combining (116) and (131), we observe that the existence of an $(L, \varepsilon)_{\max}$-code (F, G) implies that

$$L \ge \mathbb{E}[\langle \lfloor \log \varsigma_Y^{-1}(X) \rfloor \mid Y \rangle_\varepsilon], \quad (132)$$

which corresponds to the converse bound of Lemma 1.

Finally, we shall show the existence of an $(L, \varepsilon)_{\max}$-code meeting the equality in (132). In fact, constructing a variable-length stochastic code $(F_{\sup}^*, g^*)$ so that

$$F_{\sup}^*(x, Y) := \begin{cases} \boldsymbol{b}_{\varsigma_Y^{-1}(x)} & \text{if } 1 \le \varsigma_Y^{-1}(x) \le \kappa_Y, \\ B_{\sup} & \text{if } \varsigma_Y^{-1}(x) = \kappa_Y + 1, \\ \varnothing & \text{if } \kappa_Y < \varsigma_Y^{-1}(x) < \infty, \end{cases} \quad (133)$$

$$g^*(\boldsymbol{b}, Y) := x \quad \text{if } \boldsymbol{b} = \boldsymbol{b}_{\varsigma_Y^{-1}(x)} \text{ for some } x \in \mathcal{X}, \quad (134)$$

where $B_{\sup}$ denotes a $\{0, 1\}^*$-valued r.v. satisfying the condition that it is conditionally independent of X given Y (i.e., that $B_{\sup} \perp\!\!\!\perp X \mid Y$) and

$$\mathbb{P}\{B_{\sup} = \varnothing \mid Y\} = 1 - \mathbb{P}\{B_{\sup} = \boldsymbol{b}_{\kappa_Y+1} \mid Y\} = 1 - \gamma_Y \quad (135)$$

a.s., we readily see that

$$\mathbb{E}[\ell(F_{\sup}^*(X,Y))] = \mathbb{E}[\langle \lfloor \log \varsigma_Y^{-1}(X) \rfloor \mid Y \rangle_\varepsilon] \quad (136)$$

and

$$\mathbb{P}\{X \ne g^*(F_{\sup}^*(X,Y), Y) \mid Y\} = \varepsilon \quad \text{(a.s.).} \quad (137)$$

This completes the proof of Lemma 1. ∎

APPENDIX D
PROOF OF LEMMA 2

Define the event

$$C_k := \left\{ \lfloor \log x \rfloor \le \log \frac{1}{P_{X|Y}(\varsigma_Y(x) \mid Y)} \text{ for all } 1 \le x \le k \right\} \quad (138)$$

for each integer $k \ge 1$. Since ς_Y rearranges the probability masses in $P_{X|Y}(\cdot \mid Y)$ in non-increasing order (see (46)), similar to [6, Theorem 2], it can be verified by induction that

$$\mathbb{P}(C_k) = 1 \quad (139)$$

for every $k \ge 1$. Hence, the monotonicity $C_1 \supset C_2 \supset C_3 \supset \cdots$ implies that

$$\mathbb{P}\left\{ \lfloor \log \sigma_Y^{-1}(x) \rfloor \le \log \frac{1}{P_{X|Y}(x \mid Y)} \text{ for all } x \in \mathcal{X} \right\} = 1. \quad (140)$$

yielding that

$$\mathbb{P}\left\{ \log \frac{1}{P_{X|Y}(X \mid Y)} \le t \mid Y \right\} \le \mathbb{P}\left\{ \lfloor \log \sigma_Y^{-1}(X) \rfloor \le t \mid Y \right\} \quad (141)$$

a.s. for all $t > 0$. Therefore, it follows from (34) of Proposition 1 that

$$\mathbb{E}[\langle \lfloor \log \varsigma_Y^{-1}(X) \rfloor \mid Y \rangle_\varepsilon] \le \mathfrak{C}_c^\varepsilon(X \mid Y). \quad (142)$$

Thus, it follows from Lemma 1 that the left-hand inequality of (48) holds.

On the other hand, we observe that (143) written in the top of the next page holds, where

- (a) follows from (29) of Proposition 1,
- (b) follows from (29) and (34) of Proposition 1 and (141),
- (c) follows from the fact that the same argument as [4] proves

$$\mathbb{E}[\lfloor \log \varsigma_Y^{-1}(X) \rfloor \mid Y]$$
$$\ge \mathcal{H}(X \mid Y) - \log(\mathcal{H}(X \mid Y) + 1) - \log e \quad (144)$$

a.s., which leads together with Jensen's inequality that

$$\mathbb{E}[\lfloor \log \varsigma_Y^{-1}(X) \rfloor]$$
$$\ge H(X \mid Y) - \log(H(X \mid Y) + 1) - \log e; \quad (145)$$

and
- (d) follows as in (a).

Therefore, it follows from Lemma 1 that the right-hand side of (48) holds. This completes the proof of Lemma 2. ∎

$$\mathbb{E}[\langle \lfloor \log \varsigma_Y^{-1}(X) \rfloor \mid Y \rangle_\varepsilon] \overset{(a)}{=} \mathbb{E}[\lfloor \log \varsigma_Y^{-1}(X) \rfloor] - \max_{\epsilon : \mathbb{E}[\epsilon(\lfloor \log \varsigma_Y^{-1}(X) \rfloor, Y) \mid Y] = \varepsilon \text{ (a.s.)}} \mathbb{E}[\epsilon(\lfloor \log \varsigma_Y^{-1}(X) \rfloor, Y) \lfloor \log \varsigma_Y^{-1}(X) \rfloor]$$

$$\overset{(b)}{\geq} \mathbb{E}[\lfloor \log \varsigma_Y^{-1}(X) \rfloor] - \max_{\epsilon : \mathbb{E}[\epsilon(\iota(X \mid Y), Y) \mid Y] = \varepsilon \text{ (a.s.)}} \mathbb{E}[\epsilon(\iota(X \mid Y), Y) \iota(X \mid Y)]$$

$$\overset{(c)}{\geq} \mathbb{E}[\iota(X \mid Y)] - \max_{\epsilon : \mathbb{E}[\epsilon(\iota(X \mid Y), Y) \mid Y] = \varepsilon \text{ (a.s.)}} \mathbb{E}[\epsilon(\iota(X \mid Y), Y) \iota(X \mid Y)] - \log(H(X \mid Y) + 1) - \log e$$

$$\overset{(d)}{=} \mathfrak{C}_c^\varepsilon(X \mid Y) - \log(H(X \mid Y) + 1) - \log e, \tag{143}$$

APPENDIX E
PROOF OF LEMMA 3

It is clear from the definition of $\mathfrak{C}_c^\varepsilon$ stated in (27) that

$$\varepsilon = 0 \implies \mathfrak{C}_c^\varepsilon(X^n \mid Y^n) = n H(X \mid Y), \tag{146}$$
$$\varepsilon = 1 \implies \mathfrak{C}_c^\varepsilon(X^n \mid Y^n) = 0. \tag{147}$$

Hence, it suffices to consider the case where $0 < \varepsilon < 1$. It follows from (31) of Proposition 1 that

$$\mathbb{E}[\langle \iota(X^n \mid Y^n) \mid Y^n \rangle_\varepsilon \mid Y^n]$$
$$= (1 - \varepsilon) \mathcal{H}(X^n \mid Y^n)$$
$$- \int_{\eta_{Y^n}}^\infty \mathbb{P}\{\iota(X^n \mid Y^n) > t \mid Y^n\} \, dt$$
$$- \varepsilon \left(\eta_{Y^n} - \mathcal{H}(X^n \mid Y^n) \right) \quad \text{(a.s.)} \tag{148}$$

for every $n \geq 1$, where $\sigma(Y^n)$-measurable r.v.'s $\eta_{Y^n} \geq 0$ and $0 \leq \beta_{Y^n} < 1$ are given so that

$$\mathbb{P}\{\iota(X^n \mid Y^n) > \eta_{Y^n} \mid Y^n\} + \beta_{Y^n} \mathbb{P}\{\iota(X^n \mid Y^n) = \eta_{Y^n} \mid Y^n\} = \varepsilon \tag{149}$$

a.s. Similar to [5, Equations (159)–(165)], we see that (150) written in the top of the next page holds, for every $n \geq 1$, where

- (a) follows by the definition

$$b_{Y^n} := \eta_{Y^n} - \mathcal{H}(X^n \mid Y^n) - \sqrt{\mathcal{V}(X^n \mid Y^n)} \, \Phi^{-1}(1 - \varepsilon), \tag{151}$$

- (b) follows by the definition of B_{Y^n} (see (152) written in the top of the next page) with the sign function sgn : $\mathbb{R} \to \{-1, 0, 1\}$ defined by

$$\text{sgn}(u) := \begin{cases} -1 & \text{if } u < 0, \\ 0 & \text{if } u = 0, \\ 1 & \text{if } u > 0, \end{cases} \tag{153}$$

- (c) follows by the substitution rule for integrals with

$$t = r\sqrt{\mathcal{V}(X^n \mid Y^n)}, \tag{154}$$

- (d) follows by the definition of D_{Y^n} (see (155) written in the top of the next page),
- (e) follows from (82) with the trivial σ-algebra $\sigma(W) = \{\emptyset, \Omega\}$, and
- (f) follows by the definition of $f_G : [0, 1] \to [0, 1/\sqrt{2\pi}]$ stated in (4).

Substituting (150) into (148), we obtain

$$\mathbb{E}[\langle \iota(X^n \mid Y^n) \mid Y^n \rangle_\varepsilon \mid Y^n]$$
$$= (1 - \varepsilon) \mathcal{H}(X^n \mid Y^n)$$
$$- \sqrt{\mathcal{V}(X^n \mid Y^n)} \left(f_G(1 - \varepsilon) - \varepsilon \, \Phi^{-1}(1 - \varepsilon) \right)$$
$$+ B_{Y^n} - D_{Y^n} - \varepsilon \left(\eta_{Y^n} - \mathcal{H}(X^n \mid Y^n) \right)$$
$$\overset{(a)}{=} (1 - \varepsilon) \mathcal{H}(X^n \mid Y^n)$$
$$- \sqrt{\mathcal{V}(X^n \mid Y^n)} \left(f_G(1 - \varepsilon) - \varepsilon \, \Phi^{-1}(1 - \varepsilon) \right)$$
$$+ B_{Y^n} - D_{Y^n} - \varepsilon \left(b_{Y^n} + \sqrt{\mathcal{V}(X^n \mid Y^n)} \, \Phi^{-1}(1 - \varepsilon) \right)$$
$$= (1 - \varepsilon) \mathcal{H}(X^n \mid Y^n)$$
$$- \sqrt{\mathcal{V}(X^n \mid Y^n)} f_G(1 - \varepsilon) + B_{Y^n} - D_{Y^n} - \varepsilon \, b_{Y^n}, \tag{156}$$

where (a) follows by the definition of b_{Y^n} stated in (151). Taking expectations in both sides of (156), we have

$$\mathfrak{C}_c^\varepsilon(X^n \mid Y^n) = n (1 - \varepsilon) H(X \mid Y)$$
$$- \mathbb{E}\left[\sqrt{\mathcal{V}(X^n \mid Y^n)} \right] f_G(1 - \varepsilon)$$
$$+ \mathbb{E}[B_{Y^n}] + \mathbb{E}[D_{Y^n}] - \varepsilon \, \mathbb{E}[b_{Y^n}]. \tag{157}$$

Finally, we shall prove that the last three terms in (157) can be scaled as $+O(1)$ as $n \to \infty$. By Hypotheses (a) and (b) in Theorem 1, there exist two positive constants V_{inf} and T_{sup} satisfying

$$\mathcal{V}(X \mid Y) \geq V_{\text{inf}} \quad \text{(a.s.)}, \tag{158}$$
$$\mathcal{T}(X \mid Y) \leq T_{\text{sup}} \quad \text{(a.s.)}, \tag{159}$$

respectively. Using those constants, we state the following lemma.

Lemma 9. *Suppose that Hypotheses* (a) *and* (b) *in Theorem 1 hold. Given* $0 < \varepsilon < 1$, *it holds that*

$$\left| \mathbb{E}[B_{Y^n}] + \mathbb{E}[D_{Y^n}] - \varepsilon \, \mathbb{E}[b_{Y^n}] \right| \leq \frac{A (1 + \varepsilon) T_{\text{sup}}^{4/3}}{c \, V_{\text{inf}}^{3/2}} + \frac{3 \, A \, T_{\text{sup}}}{V_{\text{inf}}} \tag{160}$$

for every $n \geq n_0$, *where* $A > 0$ *is an absolute constant,* $c = c(\varepsilon) > 0$ *is a constant depending only on* ε, *and* $n_0 = n_0(\varepsilon, V_{\text{inf}}, T_{\text{sup}}) \geq 1$ *is a constant depending on* ε, V_{inf}, *and* T_{max}.

To prove Lemma 9, we shall use the following non-uniform strengthened Berry–Esseen bound.

$$\int_{\eta_{Y^n}}^{\infty} \mathbb{P}\{\iota(X^n \mid Y^n) > t \mid Y^n\}\, dt \overset{(a)}{=} \int_{b_{Y^n}}^{\infty} \mathbb{P}\Big\{\iota(X^n \mid Y^n) > \mathcal{H}(X^n \mid Y^n) + \sqrt{\mathcal{V}(X^n \mid Y^n)}\, \Phi^{-1}(1-\varepsilon) + t \;\Big|\; Y^n\Big\}\, dt$$

$$\overset{(b)}{=} \int_{0}^{\infty} \mathbb{P}\Big\{\iota(X^n \mid Y^n) > \mathcal{H}(X^n \mid Y^n) + \sqrt{\mathcal{V}(X^n \mid Y^n)}\, \Phi^{-1}(1-\varepsilon) + t \;\Big|\; Y^n\Big\}\, dt - B_{Y^n}$$

$$\overset{(c)}{=} \sqrt{\mathcal{V}(X^n \mid Y^n)} \int_{0}^{\infty} \mathbb{P}\left\{ \frac{\iota(X^n \mid Y^n) - \mathcal{H}(X^n \mid Y^n)}{\sqrt{\mathcal{V}(X^n \mid Y^n)}} > \Phi^{-1}(1-\varepsilon) + r \;\middle|\; Y^n\right\}\, dr - B_{Y^n}$$

$$= \sqrt{\mathcal{V}(X^n \mid Y^n)} \int_{\Phi^{-1}(1-\varepsilon)}^{\infty} \mathbb{P}\left\{ \frac{\iota(X^n \mid Y^n) - \mathcal{H}(X^n \mid Y^n)}{\sqrt{\mathcal{V}(X^n \mid Y^n)}} > r \;\middle|\; Y^n\right\}\, dr - B_{Y^n}$$

$$\overset{(d)}{=} \sqrt{\mathcal{V}(X^n \mid Y^n)} \int_{\Phi^{-1}(1-\varepsilon)}^{\infty} (1 - \Phi(r))\, dr - B_{Y^n} + D_{Y^n}$$

$$\overset{(e)}{=} \sqrt{\mathcal{V}(X^n \mid Y^n)} \left(\int_{\Phi^{-1}(1-\varepsilon)}^{\infty} r\, \varphi(r)\, dr - \varepsilon\, \Phi^{-1}(1-\varepsilon) \right) - B_{Y^n} + D_{Y^n}$$

$$\overset{(f)}{=} \sqrt{\mathcal{V}(X^n \mid Y^n)} \left(f_{\mathrm{G}}(1-\varepsilon) - \varepsilon\, \Phi^{-1}(1-\varepsilon) \right) - B_{Y^n} + D_{Y^n} \qquad \text{(a.s.)} \qquad (150)$$

$$B_{Y^n} := \operatorname{sgn}(b_{Y^n}) \int_{\min\{0, b_{Y^n}\}}^{\max\{0, b_{Y^n}\}} \mathbb{P}\Big\{\iota(X^n \mid Y^n) > \mathcal{H}(X^n \mid Y^n) + \sqrt{\mathcal{V}(X^n \mid Y^n)}\, \Phi^{-1}(1-\varepsilon) + t \;\Big|\; Y^n\Big\}\, dt \qquad (152)$$

$$D_{Y^n} := \sqrt{\mathcal{V}(X^n \mid Y^n)} \int_{\Phi^{-1}(1-\varepsilon)}^{\infty} \left(\mathbb{P}\left\{ \frac{\iota(X^n \mid Y^n) - \mathcal{H}(X^n \mid Y^n)}{\sqrt{\mathcal{V}(X^n \mid Y^n)}} > r \;\middle|\; Y^n\right\} - (1 - \Phi(r)) \right) dr \qquad (155)$$

Lemma 10 (non-uniform Berry–Esseen bound [34]; see also [5, Theorem 10]). *Let $n \geq 1$ be an integer, and $Z_1, Z_2, \ldots, Z_n$ independent, but not necessarily identically distributed, real-valued r.v.'s. Define the following two quantities:*

$$V_n := \sum_{i=1}^{n} \mathbb{E}[(Z_i - \mathbb{E}[Z_i])^2], \qquad (161)$$

$$T_n := \sum_{i=1}^{n} \mathbb{E}[|Z_i - \mathbb{E}[Z_i]|^3]. \qquad (162)$$

Then, it holds that

$$\left| \mathbb{P}\left\{ \sum_{i=1}^{n} (Z_i - \mathbb{E}[Z_i]) \leq z\, \sqrt{V_n} \right\} - \Phi(z) \right| \leq \frac{A\, T_n}{(1 + |z|^3)\, V_n^{3/2}} \qquad (163)$$

for every $z \in \mathbb{R}$, provided that $V_n > 0$ and $T_n < \infty$, where $A > 0$ is an absolute constant.

Note that Lemma 10 can be readily reduced to the uniform Berry–Esseen bound:

$$\sup_{z \in \mathbb{R}} \left| \mathbb{P}\left\{ \sum_{i=1}^{n} (Z_i - \mathbb{E}[Z_i]) \leq z\, \sqrt{V_n} \right\} - \Phi(z) \right| \leq \frac{A\, T_n}{V_n^{3/2}}. \qquad (164)$$

Proof of Lemma 9: Since $\iota(X^n \mid Y^n)$ is a real-valued r.v., we see that $\mathbb{P}\{\iota(X^n \mid Y^n) \leq r \mid Y^n\}$ forms a cumulative distribution function of $r \in \mathbb{R}$ a.s. (see, e.g., [35, Theorem 10.2.2]). Thus, noting that η_{Y^n} given in (149) is $\sigma(Y^n)$-measurable, it

follows from the Berry–Esseen bound stated in (164) with an absolute constant $A > 0$ that

$$\mathbb{P}\{\iota(X^n \mid Y^n) \leq \eta_{Y^n} \mid Y^n\}$$
$$\leq \Phi\left(\frac{\eta_{Y^n} - \mathcal{H}(X^n \mid Y^n)}{\sqrt{\mathcal{V}(X^n \mid Y^n)}} \right) + \frac{A\, \mathcal{T}(X^n \mid Y^n)}{\mathcal{V}(X^n \mid Y^n)^{3/2}} \quad \text{(a.s.),} \quad (165)$$
$$\mathbb{P}\{\iota(X^n \mid Y^n) < \eta_{Y^n} \mid Y^n\}$$
$$\geq \Phi\left(\frac{\eta_{Y^n} - \mathcal{H}(X^n \mid Y^n)}{\sqrt{\mathcal{V}(X^n \mid Y^n)}} \right) - \frac{A\, \mathcal{T}(X^n \mid Y^n)}{\mathcal{V}(X^n \mid Y^n)^{3/2}} \quad \text{(a.s.).} \quad (166)$$

It is clear that there exists an $n_0 = n_0(\varepsilon, V_{\inf}, T_{\sup}) \geq 1$ satisfying

$$\frac{A\, T_{\sup}}{\sqrt{n}\, V_{\inf}^{3/2}} < \frac{\min\{1 - \varepsilon, \varepsilon\}}{2} \qquad (167)$$

for every $n \geq n_0$, where note that we have assumed that $0 < \varepsilon < 1$. Since

$$\frac{A\, \mathcal{T}(X^n \mid Y^n)}{\mathcal{V}(X^n \mid Y^n)^{3/2}} = \frac{A \sum_{i=1}^{n} \mathcal{T}(X_i \mid Y_i)}{(\sum_{i=1}^{n} \mathcal{V}(X_i \mid Y_i))^{3/2}} \leq \frac{A\, T_{\sup}}{\sqrt{n}\, V_{\inf}^{3/2}} \qquad (168)$$

a.s., substituting (165) and (166) into (149), we get

$$\Phi^{-1}\left(1 - \varepsilon - \frac{A\, T_{\sup}}{\sqrt{n}\, V_{\inf}^{3/2}} \right) \leq \frac{\eta_{Y^n} - \mathcal{H}(X^n \mid Y^n)}{\sqrt{\mathcal{V}(X^n \mid Y^n)}}$$
$$\leq \Phi^{-1}\left(1 - \varepsilon + \frac{A\, T_{\sup}}{\sqrt{n}\, V_{\inf}^{3/2}} \right) \qquad (169)$$

a.s. for every $n \geq n_0$. In addition, it follows by Taylor's theorem (and the inverse function theorem) that

$$\Phi^{-1}(t+u) - \Phi^{-1}(t) = \frac{u}{f_G(s)} \tag{170}$$

for every $0 < t < 1$, every $u \in (-t, 1-t)$, and some $s \in [t, u+t]$. Applying (170) to (169), we have

$$
\begin{aligned}
|b_{Y^n}| &= \left| \eta_{Y^n} - \mathcal{H}(X^n \mid Y^n) - \sqrt{\mathcal{V}(X^n \mid Y^n)}\, \Phi^{-1}(1-\varepsilon) \right| \\
&\leq \frac{A\, T_{\text{sup}}}{c} \sqrt{\frac{\mathcal{V}(X^n \mid Y^n)}{n\, V_{\text{inf}}^3}} \\
&\leq \frac{A\, T_{\text{sup}}^{4/3}}{c\, V_{\text{inf}}^{3/2}} \qquad \text{(a.s.)}
\end{aligned}
\tag{171}
$$

for every $n \geq n_0$, where the constant $c > 0$ is given as

$$
c = c(\varepsilon) :=
\begin{cases}
f_G\!\left(\dfrac{\varepsilon}{2}\right) & \text{if } 0 < \varepsilon \leq \dfrac{1}{2}, \\[2ex]
f_G\!\left(\dfrac{1+\varepsilon}{2}\right) & \text{if } \dfrac{1}{2} < \varepsilon < 1,
\end{cases}
\tag{172}
$$

and the last inequality follows from the fact that[11]

$$
\begin{aligned}
\mathcal{V}(X^n \mid Y^n) &= \sum_{i=1}^{n} \mathcal{V}(X_i \mid Y_i) \\
&\leq \sum_{i=1}^{n} \mathcal{T}(X_i \mid Y_i)^{2/3} \\
&\leq n\, T_{\text{sup}}^{2/3} \qquad \text{(a.s.)}.
\end{aligned}
\tag{173}
$$

We readily see that F_n is bounded away from zero for sufficiently large n.

Now, it follows by the definition of B_{Y^n} stated in (152) that

$$\mathbb{E}[|B_{Y^n}|] \leq \mathbb{E}\left[\left| \int_{\min\{0, b_{Y^n}\}}^{\max\{0, b_{Y^n}\}} \mathrm{d}r \right| \right] = \mathbb{E}[|b_{Y^n}|] \tag{174}$$

for every $n \geq 1$. Moreover, it follows from (171) that

$$\mathbb{E}[|b_{Y^n}|] \leq \frac{A\, T_{\text{sup}}^{4/3}}{c\, V_{\text{inf}}^{3/2}} \tag{175}$$

for every $n \geq n_0$. Furthermore, it follows by the definition of D_{Y^n} stated in (155) that (176) written in the top of the next page holds for every $n \geq 1$, where

- (a) follows by the non-uniform Berry–Esseen theorem stated in Lemma 10 with an absolute constant $A > 0$, and
- (b) follows from (158) and (159).

Analogously, we may see that

$$\mathbb{E}[D_{Y^n}] \geq -\frac{3\, A\, T_{\text{sup}}}{V_{\text{inf}}} \tag{177}$$

for every $n \geq 1$. Combining (174)–(177), we obtain Lemma 9, as desired. ∎

The proof of Lemma 3 is completed by applying Lemma 9 to (157). ∎

---

[11]The first inequality in (173) can be verified by $(\mathbb{E}[|Z|^p \mid \mathcal{G}])^{1/p} \leq (\mathbb{E}[|Z|^q \mid \mathcal{G}])^{1/q}$ for $1 \leq p < q$ and every $\mathcal{G} \subset \mathcal{F}$.

APPENDIX F
PROOF OF LEMMA 4

To prove Lemma 4, we use the following lemma.

Lemma 11 (Bernstein's inequality; see, e.g., [36, Equation (2.10)]). *Let n be a positive integer and $Z_1, \ldots, Z_n$ real-valued and independent r.v.'s. Suppose that there exists a positive constant c satisfying $|Z_i| \leq c$ a.s. for each $1 \leq i \leq n$. Then, it holds that*

$$\mathbb{P}\left\{ \sum_{i=1}^{n}(Z_i - \mathbb{E}[Z_i]) \leq -t \right\} \leq \exp\left(-\frac{t^2}{2(\sum_{i=1}^{n} \mathbb{E}[Z_i^2] + c\,t/3)} \right) \tag{178}$$

for every positive real number t.

Assume without loss of generality that $\mathbb{P}\{\mathcal{V}(X \mid Y) > 0\} > 0$. For each integer $n \geq 1$, define the event

$$\mathcal{A}_n := \left\{ \sum_{i=1}^{n} \frac{V_c(X \mid Y) - \mathcal{V}(X_i \mid Y_i)}{\sqrt{2\, \mathbb{E}[\mathcal{V}(X \mid Y)^2]\, n \log n}} \leq 1 \right\}. \tag{179}$$

Since we have assumed that there exists a positive constant V_{sup} satisfying $\mathcal{V}(X \mid Y) \leq V_{\text{sup}}$ a.s., it follows from Lemma 11 that (180) written in the top of the next page holds, where the last equality follows from the fact that

$$V_{\text{sup}} \sqrt{\frac{2 \log n}{9\, n\, \mathbb{E}[\mathcal{V}(X \mid Y)^2]}} \leq 1 \tag{181}$$

for sufficiently large n. On the other hand, for sufficiently large n satisfying

$$\sqrt{\frac{2\, \mathbb{E}[\mathcal{V}(X \mid Y)^2] \log n}{n\, V_c(X \mid Y)^2}} \leq 1, \tag{182}$$

we observe that

$$
\begin{aligned}
&\mathbb{E}\left[\sqrt{\mathcal{V}(X^n \mid Y^n)} \right] \\
&\quad \geq \mathbb{E}\left[\sqrt{\mathcal{V}(X^n \mid Y^n)}\, \mathbf{1}_{\mathcal{A}_n} \right] \\
&\quad \geq \mathbb{E}\left[\sqrt{n\, V_c(X \mid Y)\left(1 - \sqrt{\frac{2\, \mathbb{E}[\mathcal{V}(X \mid Y)^2] \log n}{n\, V_c(X \mid Y)^2}} \right)}\, \mathbf{1}_{\mathcal{A}_n} \right] \\
&\quad \geq \sqrt{n\, V_c(X \mid Y)} \left(1 - \sqrt{\frac{2\, \mathbb{E}[\mathcal{V}(X \mid Y)^2] \log n}{n\, V_c(X \mid Y)^2}} \right) \mathbb{P}(\mathcal{A}_n),
\end{aligned}
\tag{183}
$$

where the second inequality follows by the definition of $\mathcal{A}_n$ stated in (179). Combining (51), (180), and (183), we obtain (52), completing the proof. ∎

APPENDIX G
PROOF OF LEMMA 5

Consider an $(L, \varepsilon)_{\text{avg}}$-code (F, G) satisfying

$$\mathbb{E}[\ell(F(X, Y))] \leq L, \tag{184}$$

$$\mathbb{P}\{X \neq G(F(X, Y), Y)\} \leq \varepsilon. \tag{185}$$

$$\mathbb{E}[D_{Y^n}] = \mathbb{E}\left[\sqrt{\mathcal{V}(X^n \mid Y^n)} \int_{\Phi^{-1}(1-\varepsilon)}^{\infty} \left(\mathbb{P}\left\{\frac{\iota(X^n \mid Y^n) - \mathcal{H}(X^n \mid Y^n)}{\sqrt{\mathcal{V}(X^n \mid Y^n)}} > r \mid Y^n\right\} - (1 - \Phi(r))\right) dr\right]$$

$$\overset{(a)}{\leq} A\,\mathbb{E}\left[\frac{\mathcal{T}(X^n \mid Y^n)}{\mathcal{V}(X^n \mid Y^n)}\right] \int_{\Phi^{-1}(1-\varepsilon)}^{\infty} \frac{dr}{1 + |r|^3}$$

$$\overset{(b)}{\leq} \frac{A\,T_{\sup}}{V_{\inf}} \int_{\Phi^{-1}(1-\varepsilon)}^{\infty} \frac{dr}{1 + |r|^3}$$

$$\leq \frac{2\,A\,T_{\sup}}{V_{\inf}} \int_{0}^{\infty} \frac{dr}{1 + r^3}$$

$$\leq \frac{2\,A\,T_{\sup}}{V_{\inf}} \left(\int_{0}^{1} dr + \int_{1}^{\infty} \frac{dr}{r^3}\right)$$

$$= \frac{3\,A\,T_{\sup}}{V_{\inf}} \tag{176}$$

$$\mathbb{P}(\mathcal{A}_n) \geq 1 - \exp\left(-\frac{2\,\mathbb{E}[\mathcal{V}(X \mid Y)^2]\,n\log n}{2\,n\,\mathbb{E}[\mathcal{V}(X \mid Y)^2] + (2/3)\,V_{\sup}\sqrt{2\,\mathbb{E}[\mathcal{V}(X \mid Y)^2]\,n\log n}}\right)$$

$$\geq 1 - \exp\left(-\frac{\log n}{1 + V_{\sup}\sqrt{(2\log n)/(9\,n\,\mathbb{E}[\mathcal{V}(X \mid Y)^2])}}\right)$$

$$= 1 + O\left(\frac{1}{\sqrt{n}}\right) \qquad \text{(as } n \to \infty\text{)} \tag{180}$$

As similarly done in (114) and (115) of Appendix C, construct another code (F_1, g_1) from the original code (F, G) via the random maps Φ_Y and Ψ_Y defined in (112) and (113), respectively. Obviously, the same derivations as (116) and (120) yield

$$\mathbb{E}[\ell(F_1(X, Y))] \leq L, \tag{186}$$

$$1 - \varepsilon \leq \mathbb{E}\left[\sum_{i=1}^{|\mathcal{B}(Y)|} \mathbb{P}\{X = \Psi_Y(i) \mid Y\}\right]$$

$$= \sum_{i=1}^{\infty} \mathbb{P}\{X = \Psi_Y(i)\}, \tag{187}$$

respectively, where the last equality follows from the fact that $\Psi_Y(i) = 0$ whenever $i > |\mathcal{B}(Y)|$ a.s. (see (113)). Now, defining two parameters ξ and ζ so that[12]

$$\xi := \sup\left\{k \geq 0 \mid \sum_{i=1}^{k} \mathbb{P}\{X = \Psi_Y(i)\} \leq 1 - \varepsilon\right\}, \tag{188}$$

$$\zeta := 1 - \varepsilon - \sum_{i=1}^{\xi} \mathbb{P}\{X = \Psi_Y(i)\}, \tag{189}$$

respectively. In addition, define two parameters κ and γ so that

$$\kappa := \sup\left\{k \geq 0 \mid \sum_{x=1}^{k} \mathbb{P}\{X = \varsigma_Y(x)\} \leq 1 - \varepsilon\right\}, \tag{190}$$

$$\gamma := 1 - \varepsilon - \sum_{x=1}^{\kappa} \mathbb{P}\{X = \varsigma_Y(x)\}. \tag{191}$$

[12]Note that $\alpha = 0$ if $\mathbb{P}\{X = \psi(1, Y)\} \geq 1 - \varepsilon$; and $\alpha = \infty$ if $\sum_{i=1}^{\infty} \mathbb{P}\{X = \psi(i, Y)\} = 1 - \varepsilon$ and $\mathbb{P}\{X = \psi(i, Y)\} > 0$ for all $i \geq 1$.

respectively, where ς_Y is given in (46). Furthermore, define

$$q_1(k) := \begin{cases} \mathbb{P}\{X = \Psi_Y(k)\} & \text{if } 1 \leq k \leq \xi, \\ \zeta & \text{if } k = \xi + 1, \\ 0 & \text{if } \xi + 2 \leq k < \infty, \end{cases} \tag{192}$$

$$q_2(x) := \begin{cases} \mathbb{P}\{X = \varsigma_Y(x)\} & \text{if } 1 \leq x \leq \kappa, \\ \gamma & \text{if } x = \kappa + 1, \\ 0 & \text{if } \kappa + 2 \leq x < \infty. \end{cases} \tag{193}$$

Similar to (127) and (128), we observe that

$$\mathbb{E}[\ell(F_1(X, Y))] \geq \sum_{j=1}^{\infty} \sum_{k=2^j}^{\infty} q_1(k), \tag{194}$$

$$\mathbb{E}[\langle \lfloor \log \varsigma_Y^{-1}(X) \rfloor \rangle_\varepsilon] = \sum_{j=1}^{\infty} \sum_{x=2^j}^{\infty} q_2(x). \tag{195}$$

Since ς_Y defined in (46) ensures that

$$\mathbb{P}\{X = \varsigma_Y(1)\} \geq \mathbb{P}\{X = \varsigma_Y(2)\} \geq \cdots, \tag{196}$$

it can be verified that $q_1(\cdot)$ is majorized by $q_2(\cdot)$, i.e., it follows that

$$\sum_{k=1}^{l} q_1(k) \leq \sum_{x=1}^{l} q_2(x) \qquad \text{(a.s.)} \tag{197}$$

for every $l \geq 1$, and

$$\sum_{k=1}^{\infty} q_1(k) = \sum_{x=1}^{\infty} q_2(x) = 1 - \varepsilon \qquad \text{(a.s.).} \tag{198}$$

Therefore, it follows from (194), (195), (197) and (198) that

$$\mathbb{E}[\ell(F_1(X, Y))] \geq \mathbb{E}[\langle \lfloor \log \varsigma_Y^{-1}(X) \rfloor \rangle_\varepsilon]. \tag{199}$$

Combining (186) and (199), we observe that the existence of an $(L, \varepsilon)_{\mathrm{avg}}$-code (F, G) implies that

$$L \geq \mathbb{E}[\langle \lfloor \log \varsigma_Y^{-1}(X) \rfloor \rangle_\varepsilon], \qquad (200)$$

which corresponds to the converse bound of Lemma 5.

Finally, we shall show the existence of an $(L, \varepsilon)_{\mathrm{avg}}$-code meeting the equality in (200). In fact, constructing a variable-length stochastic code $(F_{\mathrm{avg}}^*, g^*)$ so that

$$F_{\mathrm{avg}}^*(x, Y) := \begin{cases} \boldsymbol{b}_{\varsigma_Y^{-1}(x)} & \text{if } 1 \leq \varsigma_Y^{-1}(x) \leq \kappa, \\ B_{\mathrm{avg}} & \text{if } \varsigma_Y^{-1}(x) = \kappa + 1, \\ \varnothing & \text{if } \kappa < \varsigma_Y^{-1}(x) < \infty, \end{cases} \qquad (201)$$

$$g^*(\boldsymbol{b}, Y) := x \quad \text{if } \boldsymbol{b} = \boldsymbol{b}_{\varsigma_Y^{-1}(x)} \text{ for some } x \in \mathcal{X}, \qquad (202)$$

where B_{avg} denotes a $\{\varnothing, \boldsymbol{b}_{\kappa+1}\}$-valued r.v. satisfying the condition that it is independent of (X, Y) (i.e., that $B_{\mathrm{avg}} \perp\!\!\!\perp (X, Y)$), and

$$B_{\mathrm{avg}} = \begin{cases} \boldsymbol{b}_{\kappa+1} & \text{with probability } \gamma, \\ \varnothing & \text{with probability } 1 - \gamma, \end{cases} \qquad (203)$$

we readily see that

$$\mathbb{E}[\ell(F_{\mathrm{avg}}^*(X, Y))] = \mathbb{E}[\langle \lfloor \log \varsigma_Y^{-1}(X) \rfloor \rangle_\varepsilon], \qquad (204)$$

$$\mathbb{P}\{X \neq g^*(F_{\mathrm{avg}}^*(X, Y), Y)\} = \varepsilon. \qquad (205)$$

This completes the proof of Lemma 5. ∎

APPENDIX H
PROOF OF LEMMA 8

A. Proof of (74)

By Lemma 1 in Appendix D, it suffices to show that

$$-|\log c_{\mathrm{e}}| \leq N_{\max}^*(\varepsilon, X, Y) - \mathbb{E}[\langle \lfloor \log \varsigma_Y^{-1}(X) \rfloor \mid Y \rangle_\varepsilon]$$
$$\leq 1 + |\log c_{\mathrm{e}}|, \qquad (206)$$

where $N_{\max}^*(\varepsilon, X, Y) := N_{\max}^*(1, \varepsilon, X, Y)$. Throughout Appendix H, we consider one-shot ($n = 1$) guessing strategies as defined in Section IV (by taking n therein to be 1). Specifically, we now consider an $(N, \varepsilon)_{\max}$-*guessing strategy* $(g, \pi(\cdot \mid \cdot))$ satisfying

$$\mathbb{E}[\log \mathsf{G}(X, Y)] \leq N, \qquad (207)$$

$$\mathbb{P}\{\mathsf{G}(X, Y) \neq g(X, Y) \mid Y\} \leq \varepsilon \qquad (\text{a.s.}). \qquad (208)$$

It is clear that

$$\left| \mathbb{E}[(\log \mathsf{G}(X, Y)) \, \mathbf{1}_{\{\mathsf{G}(X,Y) \neq g(X,Y)\}}] \right| \leq |\log c_{\mathrm{e}}|. \qquad (209)$$

It follows from (208) that

$$1 - \varepsilon \leq 1 - \mathbb{P}\{\mathsf{G}(X, Y) \neq g(X, Y) \mid Y\}$$
$$= \mathbb{P}\{\mathsf{G}(X, Y) = g(X, Y) \mid Y\}$$
$$= \sum_{k=1}^{\infty} \mathbb{P}\{\mathsf{G}(X, Y) = k \mid Y\} \qquad (\text{a.s.}), \qquad (210)$$

where the last equality follows from the fact that $\mathsf{G}(X, Y) = k$ only if $g(X, Y) = k$. Based on (210), define the $\sigma(Y)$-measurable real-valued r.v.'s ν_Y and υ_Y as follows:

$$\nu_Y := \sup\left\{ k \geq 1 \;\middle|\; \sum_{x=1}^{k} \mathbb{P}\{\mathsf{G}(X, Y) = k \mid Y\} \leq 1 - \varepsilon \right\}, \qquad (211)$$

$$\upsilon_Y := 1 - \varepsilon - \sum_{x=1}^{\nu_Y} \mathbb{P}\{\mathsf{G}(X, Y) = k \mid Y\}, \qquad (212)$$

respectively. Furthermore, define

$$p_3(k \mid Y) := \begin{cases} \mathbb{P}\{\mathsf{G}(X, Y) = k \mid Y\} & \text{if } 1 \leq k \leq \nu_Y, \\ \upsilon_Y & \text{if } k = \nu_Y + 1, \\ 0 & \text{if } \nu_Y + 2 \leq k < \infty. \end{cases} \qquad (213)$$

Similar to (127) in Appendix C, we get

$$\mathbb{E}[\lfloor \log \mathsf{G}(X, Y) \rfloor \, \mathbf{1}_{\{\mathsf{G}(X,Y)=g(X,Y)\}} \mid Y] \geq \sum_{j=1}^{\infty} \sum_{i=2^j}^{\infty} p_3(i \mid Y) \qquad (214)$$

a.s. Moreover, in the same way as (129) and (130) in Appendix C, it can be verified that $p_3(\cdot \mid Y)$ is majorized by $p_2(\cdot \mid Y)$ a.s., where $p_2(\cdot \mid Y)$ is defined in (126). Therefore, combining (207)–(209) and (214), the existence of an $(N, \varepsilon)_{\max}$-guessing strategy implies that

$$N + |\log c_{\mathrm{e}}| \geq \mathbb{E}[\langle \lfloor \log \varsigma_Y^{-1}(X) \rfloor \mid Y \rangle_\varepsilon], \qquad (215)$$

which corresponds to the left-hand inequality of (206).

Finally, considering the guessing strategy $(g^*, \pi_{\max}^*(\cdot \mid \cdot))$ given by

$$g^*(x, Y) = \varsigma_Y^{-1}(x) \qquad (\text{a.s.}), \qquad (216)$$

$$\pi_{\max}^*(x \mid Y) = \begin{cases} 0 & \text{if } 1 \leq x \leq \kappa_Y, \\ 1 - \dfrac{\gamma_Y}{P_{X|Y}(\varsigma_Y(x) \mid Y)} & \text{if } x = \kappa_Y + 1, \\ 1 & \text{if } \kappa_Y + 2 \leq x < \infty, \end{cases} \qquad (217)$$

and denoting by $\mathsf{G}_{\max}^* : \mathcal{X} \times \mathcal{Y} \to \mathcal{X}$ the giving-up guessing function induced by the strategy $(g^*, \pi_{\max}^*(\cdot \mid \cdot))$, we obtain after some algebra that

$$\mathbb{P}\{\mathsf{G}_{\max}^*(X, Y) \neq g^*(X, Y) \mid Y\} = \varepsilon \qquad (\text{a.s.}) \qquad (218)$$

and

$$\mathbb{E}[(\log \mathsf{G}_{\max}^*(X, Y)) \, \mathbf{1}_{\{\mathsf{G}(X,Y)=g(X,Y)\}}]$$
$$\leq \mathbb{E}[\lfloor \log \mathsf{G}_{\max}^*(X, Y) \rfloor \, \mathbf{1}_{\{\mathsf{G}(X,Y)=g(X,Y)\}}] + 1$$
$$= \mathbb{E}[\langle \lfloor \log \varsigma_Y^{-1}(X) \rfloor \mid Y \rangle_\varepsilon] + 1, \qquad (219)$$

which, together with (209) and (214), imply the right-hand inequality of (206). This completes the proof of (74). ∎

B. Proof of (75)

By Lemma 5, it suffices to show that

$$-|\log c_{\mathrm{e}}| \le N_{\mathrm{avg}}^*(\varepsilon, X, Y) - \mathbb{E}[\langle \lfloor \log \varsigma_Y^{-1}(X) \rfloor \mid Y \rangle_\varepsilon]$$
$$\le 1 + |\log c_{\mathrm{e}}|, \tag{220}$$

where $N_{\mathrm{avg}}^*(\varepsilon, X, Y) \coloneqq N_{\mathrm{avg}}^*(1, \varepsilon, X, Y)$. Consider an $(N, \varepsilon)_{\mathrm{avg}}$-*guessing strategy* $(\mathsf{g}, \pi(\cdot \mid \cdot))$ satisfying

$$\mathbb{E}[\log \mathsf{G}(X, Y)] \le N, \tag{221}$$
$$\mathbb{P}\{\mathsf{G}(X, Y) \ne \mathsf{g}(X, Y)\} \le \varepsilon, \tag{222}$$

where $\mathsf{G} : \mathcal{X} \times \mathcal{Y} \to \mathcal{X}$ is the giving-up guessing function induced by the strategy $(\mathsf{g}, \pi(\cdot \mid \cdot))$. Similar to (210), one has

$$1 - \varepsilon \le \sum_{k=1}^{\infty} \mathbb{P}\{\mathsf{G}(X, Y) = k\}. \tag{223}$$

Based on (223), define two parameters ν and υ by

$$\nu \coloneqq \sup\left\{ k \ge 1 \;\middle|\; \sum_{x=1}^{k} \mathbb{P}\{\mathsf{G}(X, Y) = k\} \le 1 - \varepsilon \right\}, \tag{224}$$

$$\upsilon \coloneqq 1 - \varepsilon - \sum_{x=1}^{\nu} \mathbb{P}\{\mathsf{G}(X, Y) = k\}, \tag{225}$$

respectively. Moreover, define

$$q_3(k) \coloneqq \begin{cases} \mathbb{P}\{\mathsf{G}(X, Y) = k\} & \text{if } 1 \le k \le \nu, \\ \upsilon & \text{if } k = \nu + 1, \\ 0 & \text{if } \nu + 2 \le k < \infty, \end{cases} \tag{226}$$

Similar to (214), we observe that

$$\mathbb{E}[\lfloor \log \mathsf{G}(X, Y) \rfloor \, \mathbf{1}_{\{\mathsf{G}(X,Y)=\mathsf{g}(X,Y)\}}] \ge \sum_{j=1}^{\infty} \sum_{k=2^j}^{\infty} q_3(k), \tag{227}$$

Moreover, in the same way as (197) and (198) in Appendix G, it can be verified that $q_3(\cdot)$ is majorized by $q_2(\cdot)$, where $q_2(\cdot)$ is defined in (193). Therefore, combining (209), (221), (222), and (227), the existence of an $(N, \varepsilon)_{\mathrm{avg}}$-guessing strategy implies that

$$N + |\log c_{\mathrm{e}}| \ge \mathbb{E}[\langle \lfloor \log \varsigma_Y^{-1}(X) \rfloor \rangle_\varepsilon], \tag{228}$$

which corresponds to the left-hand inequality of (220).

Finally, considering the guessing strategy $(\mathsf{g}^*, \pi_{\mathrm{avg}}^*(\cdot \mid \cdot))$ so that g^* is given as (216) and

$$\pi_{\mathrm{avg}}^*(x \mid Y) = \begin{cases} 0 & \text{if } 1 \le x \le \kappa, \\ 1 - \dfrac{\gamma}{\mathbb{P}\{X = \varsigma_Y(x)\}} & \text{if } x = \kappa + 1, \\ 1 & \text{if } \kappa + 2 \le x < \infty, \end{cases} \tag{229}$$

and denoting by $\mathsf{G}_{\mathrm{avg}}^* : \mathcal{X} \times \mathcal{Y} \to \mathcal{X}$ the giving-up guessing function induced by the strategy $(\mathsf{g}^*, \pi_{\mathrm{avg}}^*(\cdot \mid \cdot))$, we obtain after some algebra that

$$\mathbb{P}\{\mathsf{G}_{\mathrm{avg}}^*(X, Y) \ne \mathsf{g}^*(X, Y)\} = \varepsilon \tag{230}$$

and

$$\mathbb{E}[(\log \mathsf{G}_{\mathrm{avg}}^*(X, Y)) \, \mathbf{1}_{\{\mathsf{G}(X,Y)=\mathsf{g}(X,Y)\}}]$$
$$\le \mathbb{E}[(\lfloor \log \mathsf{G}_{\mathrm{avg}}^*(X, Y) \rfloor) \, \mathbf{1}_{\{\mathsf{G}(X,Y)=\mathsf{g}(X,Y)\}}] + 1$$
$$= \mathbb{E}[\langle \lfloor \log \varsigma_Y^{-1}(X) \rfloor \rangle_\varepsilon] + 1, \tag{231}$$

which, together with (209) and (227), imply the right-hand inequality of (220). This completes the proof of (75). ∎

APPENDIX I
RELAXATIONS OF HYPOTHESES IN THEOREM 1

A. Proof of Proposition 2

Suppose that $\mathbb{P}\{X \in \mathcal{A}\} = 1$ for some finite $\mathcal{A} \subset \mathcal{X}$. Then, we readily see that

$$\sum_{x \in \mathcal{A}} P_{X|Y}(x \mid Y) = 1 \qquad \text{(a.s.)}. \tag{232}$$

Hence, the well-known upper bound on the Shannon entropy shows that

$$\sum_{x \in \mathcal{X}} P_{X|Y}(x \mid Y) \log \frac{1}{P_{X|Y}(x \mid Y)}$$
$$= \sum_{x \in \mathcal{A}} P_{X|Y}(x \mid Y) \log \frac{1}{P_{X|Y}(x \mid Y)}$$
$$\le \log|\mathcal{A}| \qquad \text{(a.s.)}. \tag{233}$$

Therefore, it can be verified by the dominated convergence theorem for the conditional expectation that

$$\mathcal{H}(X \mid Y) \le \log|\mathcal{A}| \qquad \text{(a.s.)}. \tag{234}$$

On the other hand, we get

$$\sum_{x \in \mathcal{A}} P_{X|Y}(x \mid Y) \left| \log \frac{1}{P_{X|Y}(x \mid Y)} - \mathcal{H}(X \mid Y) \right|^3$$
$$\le \sum_{x \in \mathcal{A}} P_{X|Y}(x \mid Y) \log^3 \frac{1}{P_{X|Y}(x \mid Y)} + \mathcal{H}(X \mid Y)^3. \tag{235}$$

Since the mapping $u \mapsto u \ln^3(1/u)$ on $[0, 1]$ is maximized at $u = \mathrm{e}^{-3}$, it follows from (232) and (235) that

$$\sum_{x \in \mathcal{A}} P_{X|Y}(x \mid Y) \log^3 \frac{1}{P_{X|Y}(x \mid Y)} \le \frac{27 |\mathcal{A}|}{(\mathrm{e} \ln 2)^3} \tag{236}$$

a.s., where $\ln$ stands for the natural logarithm. Combining (234)–(236), we can obtain from the dominated convergence theorem that

$$\mathcal{T}(X \mid Y) \le \frac{27 |\mathcal{A}|}{(\mathrm{e} \ln 2)^3} + \log^3 |\mathcal{A}| \qquad \text{(a.s.)}, \tag{237}$$

which implies that $\mathcal{T}(X \mid Y)$ is bounded away from infinity a.s. This completes the proof of Proposition 2. ∎

B. Proof of Proposition 3

Assume without loss of generality that $P_Y(y)$ is positive for each $y \in \mathcal{Y}$. Hypothesis (a) in Theorem 1 is used only to ensure Lemma 3; thus, it suffices to prove Lemma 3 without Hypothesis (a) in Theorem 1.

Let $\eta(y) \ge 0$ be a real number satisfying

$$\mathbb{P}\{\iota(X \mid Y) > \eta(y) \mid Y = y\}$$
$$+ \beta(y) \mathbb{P}\{\iota(X \mid Y) = \eta(y) \mid Y = y\} = \varepsilon \tag{238}$$

for some $0 \le \beta(y) < 1$. Define

$$\eta_{\max} = \max_{y \in \mathcal{Y}} \eta(y). \tag{239}$$

Since we now do not assume that Hypothesis (a) in Theorem 1 holds, there may exist a $y \in \mathcal{Y}$ satisfying $\mathcal{V}(X \mid y) = 0$. If $\mathcal{V}(X \mid y) = 0$ for every $y \in \mathcal{Y}$, then we readily see that

$$\mathbb{E}[\langle \iota(X^n \mid Y^n) \mid Y^n \rangle_\varepsilon \mid Y^n = \boldsymbol{y}] = (1 - \varepsilon) \sum_{i=1}^n \mathcal{H}(X \mid y_i) \tag{240}$$

for every $n \geq 1$ and $\boldsymbol{y} \in \mathcal{Y}^n$. Therefore, Lemma 3 also holds even if $V(X \mid y) = 0$ for every $y \in \mathcal{Y}$.

In the following, we assume that there exists a $y \in \mathcal{Y}$ satisfying $V(X \mid y) > 0$. Define

$$H_{\max} := \max_{y \in \mathcal{Y}} \mathcal{H}(X \mid y), \tag{241}$$

$$V_{\max} := \max_{y \in \mathcal{Y}} \mathcal{V}(X \mid y), \tag{242}$$

$$V_{\min} := \min_{\substack{y \in \mathcal{Y}: \\ \mathcal{V}(X \mid y) > 0}} \mathcal{V}(X \mid y), \tag{243}$$

$$T_{\max} := \max_{y \in \mathcal{Y}} \mathcal{T}(X \mid y). \tag{244}$$

Note that the three numbers $H_{\max}$, $V_{\max}$, and $T_{\max}$ are finite since Hypothesis (b) in Theorem 1 is satisfied. To prove Lemma 3 without Hypothesis (a) in Theorem 1, it suffices to prove an analog of Lemma 9 stated in Appendix E without Hypothesis (a) in Theorem 1. More precisely, we shall prove the following lemma.

Lemma 12. *Recall that b_{Y^n}, B_{Y^n}, and D_{Y^n} are defined in* (151), (152), *and* (155), *respectively (see Appendix E). Suppose that the following hold: (i) $\mathcal{Y}$ is a finite alphabet, (ii) there exists a $y \in \mathcal{Y}$ satisfying $V(P_{X|Y=y}) > 0$, and (iii) Hypothesis* (b) *in Theorem 1 holds. Given $0 < \varepsilon < 1$, it holds that*

$$\left| \mathbb{E}[B_{Y^n}] + \mathbb{E}[D_{Y^n}] - \varepsilon \mathbb{E}[b_{Y^n}] \right|$$

$$\leq (1 + \varepsilon) \left(\frac{A T_{\max}^{4/3}}{c V_{\min}^{3/2}} + K \left(\eta_{\max} + H_{\max} + V_{\max} \Phi^{-1}(1 - \varepsilon) \right) \right)$$

$$+ \frac{3 A T_{\max}}{V_{\min}} \tag{245}$$

for every $n \geq K$, where $A > 0$ is an absolute constant, $c = c(\varepsilon) > 0$ is a constant depending only on ε, and $K = K(\varepsilon, V_{\min}, T_{\max}) > 0$ is a constant depending on ε, $V_{\min}$, and $T_{\max}$.

Proof of Lemma 12: Fix an infinite sequence $\boldsymbol{y} = (y_1, y_2, \dots) \in \mathcal{Y}^{\mathbb{N}}$ arbitrarily. For each $n \geq 1$, denote by $\boldsymbol{y}^{(n)} = (y_1, \dots, y_n)$ the n-length prefix of $\boldsymbol{y}$. For each $n \geq 1$, consider two parameters $\eta_n \geq 0$ and $0 \leq \beta_n < 1$ given so that

$$\mathbb{P}\{\iota(X^n \mid Y^n) > \eta_n \mid Y^n = \boldsymbol{y}^{(n)}\}$$

$$+ \beta_n \mathbb{P}\{\iota(X^n \mid Y^n) = \eta_n \mid Y^n = \boldsymbol{y}^{(n)}\} = \varepsilon. \tag{246}$$

Let $\mathcal{K} \subset \mathbb{N}$ be the subset in which for all $k \in \mathcal{K}$, there exists a finite $\mathcal{A}_k \subset \mathcal{X}$ satisfying $P_{X|Y}(x \mid y_k) = 1/|\mathcal{A}_k|$ for each $x \in \mathcal{A}_k$. Define $k(n) := |\{1 \leq k \leq n \mid k \notin \mathcal{K}\}|$ for each $n \geq 1$. Moreover, let $n_1 \geq 1$ be chosen so that

$$n_1 := \sup \left\{ n \geq 1 \ \middle| \ \frac{A T_{\max}}{\sqrt{k(n) + 1} \, V_{\min}} \geq \frac{\min\{1 - \varepsilon, \varepsilon\}}{2} \right\}. \tag{247}$$

Since $\mathcal{V}(X \mid y_k) = \mathcal{T}(X \mid y_k) = 0$ for every $k \in \mathcal{K}$, and since $\mathcal{V}(X \mid y_i) \mathcal{T}(X \mid y_i) > 0$ for every $i \in \mathbb{N} \setminus \mathcal{K}$, we observe that

$$\frac{A \mathcal{T}(X^n \mid \boldsymbol{y}^{(n)})}{\mathcal{V}(X^n \mid \boldsymbol{y}^{(n)})^{3/2}} = \frac{A \sum_{i=1:i \notin \mathcal{K}}^n \mathcal{T}(X \mid y_i)}{(\sum_{i=1:i \notin \mathcal{K}}^n \mathcal{V}(X \mid |y_i))^{3/2}}$$

$$\leq \frac{A T_{\max}}{\sqrt{k(n)} \, V_{\min}^{3/2}}; \tag{248}$$

therefore, it can be shown by the same way as (171) that

$$\left| \eta_n - \mathcal{H}(X^n \mid \boldsymbol{y}^{(n)}) - \sqrt{\mathcal{V}(X^n \mid \boldsymbol{y}^{(n)})} \, \Phi^{-1}(1 - \varepsilon) \right| \leq \frac{A T_{\max}^{4/3}}{c V_{\min}^{3/2}} \tag{249}$$

for every $n \geq n_1$, provided that $n_1 < \infty$, where $A > 0$ is an absolute constant that appears in Lemma 10, and $c = c(\varepsilon) > 0$ a constant is given in (172).

Now, consider the case where $n_1 = \infty$. Note that $n_1 = \infty$ if and only if

$$\lim_{n \to \infty} k(n) \leq \left(\frac{2 A T_{\max}}{V_{\min} \min\{1 - \varepsilon, \varepsilon\}} \right)^2 - 1 =: K(\varepsilon, V_{\min}, T_{\max}). \tag{250}$$

It is clear from the definition of $\mathcal{K}$ that

$$\mathcal{H}(X^n \mid \boldsymbol{y}^{(n)}) = \sum_{i=1:i \notin \mathcal{K}}^n \mathcal{H}(X \mid y_i) + \sum_{j=1:j \in \mathcal{K}}^n \log |\mathcal{A}_j| \tag{251}$$

for each $n \geq 1$. If $1 \in \mathcal{K}$, then

$$\mathbb{P}\{\iota(X_1 \mid Y_1) > t \mid Y_1 = y_1\} = \begin{cases} 0 & \text{if } t \leq \log |\mathcal{A}_1|, \\ 1 & \text{if } t > \log |\mathcal{A}_1|, \end{cases} \tag{252}$$

implying that

$$\eta_1 = \log |\mathcal{A}_1|. \tag{253}$$

Moreover, for each $k \in \mathcal{K}$ satisfying $k \geq 2$, we see that

$$\mathbb{P}\{\iota(X^k \mid Y^k) > \eta_k \mid Y^k = \boldsymbol{y}^{(n)}\}$$

$$= \mathbb{P}\{\iota(X^{k-1} \mid Y^{k-1}) > \eta_{k-1} + \log |\mathcal{A}| \mid Y^k = \boldsymbol{y}^{(n)}\} \tag{254}$$

for every $t \in \mathbb{R}$, implying that

$$\eta_k = \eta_{k-1} + \log |\mathcal{A}_k|. \tag{255}$$

Therefore, since $\mathcal{V}(X \mid y_k) = 0$ for each $k \in \mathcal{K}$, it follows from (250), (251), (253), and (254) that

$$\left| \eta_n - \mathcal{H}(X^n \mid \boldsymbol{y}^{(n)}) - \sqrt{\mathcal{V}(X^n \mid \boldsymbol{y}^{(n)})} \, \Phi^{-1}(1 - \varepsilon) \right|$$

$$\leq K(\varepsilon, V_{\min}, T_{\max}) \left(\eta_{\max} + H_{\max} + V_{\max} \Phi^{-1}(1 - \varepsilon) \right) \tag{256}$$

for every $n \geq 1$, provided that $n_1 = \infty$.

Combining (249) and (256), we obtain

$$\left| \eta_n - \mathcal{H}(X^n \mid \boldsymbol{y}^{(n)}) - \sqrt{\mathcal{V}(X^n \mid \boldsymbol{y}^{(n)})} \, \Phi^{-1}(1 - \varepsilon) \right|$$

$$\leq \frac{A T_{\max}^{4/3}}{c V_{\min}^{3/2}} + K(\varepsilon, V_{\min}, T_{\max}) \left(\eta_{\max} + H_{\max} + V_{\max} \Phi^{-1}(1 - \varepsilon) \right) \tag{257}$$

for every $n \geq K(\varepsilon, V_{\min}, T_{\max})$. Therefore, since the infinite sequence $\boldsymbol{y} = (y_1, y_2, \dots)$ is arbitrary, we have

$$\mathbb{E}[|b_{Y^n}|] \leq \frac{A\, T_{\max}^{4/3}}{c\, V_{\min}^{3/2}}$$
$$+ K(\varepsilon, V_{\min}, T_{\max}) \left(\eta_{\max} + H_{\max} + V_{\max}\, \Phi^{-1}(1-\varepsilon) \right) \quad (258)$$

for every $n \geq K(\varepsilon, V_{\min}, T_{\max})$. The proof of Lemma 12 is finally completed by combining (174), (176), (177), and (258). ∎

Lemma 3 without Hypothesis (a) in Theorem 1 is now ensured by applying Lemma 12 to (157). This completes the proof of Proposition 3. ∎

APPENDIX J
A COUNTEREXAMPLE OF
CONVERSE PART IN [5, SECTION II-A]

In this appendix, we consider a variable-length compression problem for a discrete memoryless source in the *absence* of side-information. Here, we use similar notations to those in [5]. Fix a real number $0 < \epsilon < 1$. Suppose that the source S defined as a positive-integer-valued r.v. satisfies that $P_S(1) \geq P_S(2) \geq \cdots$. As written in Remark 2, consider a stochastic encoder $P_{W|S}$ defined as a conditional probability distribution of S given a $\{0,1\}^*$-valued r.v. W. In addition, consider a deterministic decoder $\mathsf{g} : \{0,1\}^* \to \{1,2,\dots\}$.

In [5, Section II-A], it was stated that an optimal variable-length stochastic code $(P_{W|S}^*, \mathsf{g}^*)$ like (133) and (134), or like (201) and (202), has an average codeword length and error probability that are both no larger than those of the given variable-length stochastic code $(P_{W|S}, \mathsf{g})$. This is an unconditional version of Lemmas 1 and 5. In the particular case of S satisfying $P_S(1) + P_S(2) + \cdots + P_S(M) = 1 - \epsilon$ for some $M \geq 1$, its proof is written in Lines 3–19 of the left column of Page 4318 of [5, Section II-A] as follows:

"*Formally, for a given encoder $P_{W|S}$, the optimal decoder is always deterministic and we denote it by g. Consider* [sic] $w_0 \in \{0,1\}^* \backslash \varnothing$ *and source realization m with $P_{W|S=m}(w_0) > 0$. If $\mathsf{g}(w_0) \neq m$, the average length can be decreased, without affecting the probability of error, by setting $P_{W|S=m}(w_0) = 0$ and adjusting $P_{W|S=m}(\varnothing)$ accordingly. This argument implies that the optimal encoder has at most one source realization m mapping to each $w_0 \neq \varnothing$. Next, let $m_0 = \mathsf{g}(\varnothing)$ and by a similar argument conclude that $P_{W|S=m_0}(\varnothing) = 1$. But then, interchanging m_0 and 1 leads to the same or better probability of error and shorter average length, which implies that the optimal encoder maps 1 to $\varnothing$. Continuing in the same manner for $m_0 = \mathsf{g}(0), \mathsf{g}(1), \dots, \mathsf{g}(\mathsf{f}_S^*(M))$, we conclude that the optimal code maps $\mathsf{f}(m) = \mathsf{f}_S^*(m)$, $m = 1, \dots, M$. Finally, assigning the remaining source outcomes whose total mass is ϵ to $\varnothing$ shortens the average length without affecting the error probability, so $\mathsf{f}(m) = \varnothing$, $m > M$ is optimal.*"

In the above proof, note that $\mathsf{f} : \{1,2,\dots\} \to \{0,1\}^*$ denotes a deterministic encoder induced by the stochastic encoder $P_{W|S}$ constructed by the above procedures, and $\mathsf{f}_S^* : \{1,2,\dots\} \to \{0,1\}^*$ stands for the optimal one-to-one code satisfying $\mathsf{f}_S^*(m) = \boldsymbol{b}_m$ for every $m \geq 1$ (cf. [4]), where $\{\boldsymbol{b}_i\}_{i=1}^\infty$

denotes the lexicographical ordering of the strings in $\{0,1\}^*$ (cf. Appendix C).

One can, however, show a counterexample of the triplet $(X, P_{B|X}, \mathsf{g})$ in which the above proof strategy fails to prove the desired assertion. Let $\epsilon = 1/6$ be fixed, and P_S the source distribution given as

$$P_S(m) = \begin{cases} \dfrac{1}{2} & \text{if } m = 1, \\ \dfrac{1}{3} & \text{if } m = 2, \\ \dfrac{1}{6} & \text{if } m = 3, \\ 0 & \text{otherwise.} \end{cases} \quad (259)$$

Since $P_S(1) + P_S(2) = 1 - \epsilon$, it is clear that $M = 2$ in this case. Consider the stochastic encoder $P_{W|S}$ defined as

$$P_{W|S=m}(w) = \begin{cases} \dfrac{5}{6} & \text{if } m = 1 \text{ and } w = 0, \\ \dfrac{1}{6} & \text{if } m = 1 \text{ and } w = 1 \\ 1 & \text{if } m = 2 \text{ and } w = \varnothing, \\ \dfrac{1}{2} & \text{if } m = 3 \text{ and } w \in \{0,1\}, \\ 0 & \text{otherwise,} \end{cases} \quad (260)$$

and the decoder g given as

$$\mathsf{g}(w) = \begin{cases} 1 & \text{if } w = 0, \\ 2 & \text{if } w = \varnothing, \\ 3 & \text{if } w = 1, \\ \text{arbitrary} & \text{otherwise.} \end{cases} \quad (261)$$

Then, the error probability and the average codeword length can be calculated as $\mathbb{P}\{S \neq \hat{S}\} = 1/6 = \epsilon$ and $\mathbb{E}[\ell(W)] = 2/3$, respectively, where $\hat{S} = \mathsf{g}(W)$. By applying the above proof strategy to the given variable-length stochastic code $(P_{W|S}, \mathsf{g})$, we construct the optimal encoder and decoder:

$$P_{W|S=m}^*(w) = \begin{cases} 1 & \text{if } m = 1 \text{ and } w = \varnothing, \\ 1 & \text{if } m = 2 \text{ and } w = 0, \\ 1 & \text{if } m = 3 \text{ and } w = \varnothing, \\ 0 & \text{otherwise,} \end{cases} \quad (262)$$

$$\mathsf{g}^*(w) = \begin{cases} 1 & \text{if } w = \varnothing, \\ 2 & \text{if } w = 0, \\ \text{arbitrary} & \text{otherwise,} \end{cases} \quad (263)$$

respectively, yielding the optimal performance: $\mathbb{P}\{S \neq \hat{S}\} = 1/6 = \epsilon$ and $\mathbb{E}[\ell(W)] = 1/3$.

By the procedure "*Consider* [sic] $w_0 \in \{0,1\}^* \setminus \varnothing$ *and source realization m with $P_{W|S=m}(w_0) > 0$. If $\mathsf{g}(w_0) \neq m$, the average length can be decreased, without affecting the probability of error, by setting $P_{W|S=m}(w_0) = 0$ and adjusting $P_{W|S=m}(\varnothing)$ accordingly,*" we modify the stochastic encoder $P_{W|S}$ to another $P_{W|S}^{(1)}$ so that $P_{W|S=m}^{(1)}(w_0) = 0$ if $w_0 \neq \varnothing$ and

$\mathsf{g}(w_0) \neq m$; and $P_{W|S=m}^{(1)}(w_0) = P_{W|S=m}(w_0)$ if $w_0 \neq \varnothing$ and $\mathsf{g}(w_0) = m$; and

$$P_{W|S=m}^{(1)}(\varnothing) = P_{W|S=m}(\varnothing)$$
$$+ \sum_{\substack{w \in \{0,1\}^* \setminus \{\varnothing\}: \\ \mathsf{g}(w) \neq m}} P_{W|S=m}(w) \qquad (264)$$

for every $m \geq 1$. Namely, we see from (260) that

$$P_{W|S=m}^{(1)}(w) = \begin{cases} \dfrac{5}{6} & \text{if } m = 1 \text{ and } w = 0, \\ \dfrac{1}{6} & \text{if } m = 1 \text{ and } w = \varnothing, \\ 1 & \text{if } m = 2 \text{ and } w = \varnothing, \\ \dfrac{1}{2} & \text{if } m = 3 \text{ and } w \in \{\varnothing, 1\}, \\ 0 & \text{otherwise.} \end{cases} \qquad (265)$$

As mentioned as "*the average length can be decreased, without affecting the probability of error*," the error probability and the average codeword length are then given as $\mathbb{P}\{S \neq \hat{S}\} = 1/6 = \epsilon$ and $\mathbb{E}[\ell(W)] = 1/2$, respectively.

Next, consider the procedure "*Next, let $m_0 = \mathsf{g}(\varnothing)$ and by a similar argument conclude that $P_{W|S=m_0}(\varnothing) = 1$. But then, interchanging m_0 and 1 leads to the same or better probability of error and shorter average length, which implies that the optimal encoder maps 1 to $\varnothing$.*" In this example, we see from (261) that $m_0 = 2$, and interchanging the source symbols $m_0 = 2$ and 1 results in the following encoder and decoder:

$$P_{W|S=m}^{(2)}(w) = \begin{cases} 1 & \text{if } m = 1 \text{ and } w = \varnothing, \\ \dfrac{5}{6} & \text{if } m = 2 \text{ and } w = 0, \\ \dfrac{1}{6} & \text{if } m = 2 \text{ and } w = \varnothing, \\ \dfrac{1}{2} & \text{if } m = 3 \text{ and } w \in \{\varnothing, 1\}, \\ 0 & \text{otherwise,} \end{cases} \qquad (266)$$

$$\mathsf{g}^{(2)}(w) = \begin{cases} 1 & \text{if } w = \varnothing, \\ 2 & \text{if } w = 0, \\ 3 & \text{if } w = 1, \\ \text{arbitrary} & \text{otherwise,} \end{cases} \qquad (267)$$

respectively, and we readily see that $\mathbb{P}\{S \neq \hat{S}\} = 5/36 < \epsilon$ and $\mathbb{E}[\ell(W)] = 13/36$.

Moreover, consider the iterative procedure "*Continuing in the same manner for $m_0 = \mathsf{g}(0), \mathsf{g}(1), \ldots, \mathsf{g}(\mathsf{f}_S^*(M))$, we conclude that the optimal code maps $\mathsf{f}(m) = \mathsf{f}_S^*(m)$, $m = 1, \ldots, M$.*" However, since $M = 2$ and $\mathsf{g}^{(2)} \circ \mathsf{f}_S^*(2) = 2$, this iterative procedure does not change the variable-length stochastic code $(P_{W|S}^{(2)}, \mathsf{g}^{(2)})$ at all. Namely, since the stochastic encoder $P_{W|S}^{(2)}$ given in (266) stochastically maps the source symbols $m = 1$ and $m = 3$, we cannot conclude that "*the optimal code maps $\mathsf{f}(m) = \mathsf{f}_S^*(m)$, $m = 1, \ldots, M$*" in this example.

Finally, consider the procedure "*Finally, assigning the remaining source outcomes whose total mass is ϵ to $\varnothing$ shortens the average length without affecting the error probability, so*

$\mathsf{f}(m) = \varnothing$, $m > M$ *is optimal.*" Then, we have the following encoder:

$$P_{W|S=m}^{(3)}(w) = \begin{cases} 1 & \text{if } m = 1 \text{ and } w = \varnothing, \\ \dfrac{5}{6} & \text{if } m = 2 \text{ and } w = 0, \\ \dfrac{1}{6} & \text{if } m = 2 \text{ and } w = \varnothing, \\ 1 & \text{if } m = 3 \text{ and } w = \varnothing, \\ 0 & \text{otherwise,} \end{cases} \qquad (268)$$

and we observe from the variable-length stochastic code $(P_{W|S}^{(3)}, \mathsf{g}^{(2)})$ that $\mathbb{P}\{S \neq \hat{S}\} = 2/9 > \epsilon$ and $\mathbb{E}[\ell(W)] = 5/18$. This calculation, however, violates the error probability constraint $\mathbb{P}\{S \neq \hat{S}\} \leq \epsilon$, which justifies our claim that there is an error in the proof in [5, Section II-A].

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