A unified framework for deterministic and probabilistic $\mathcal{D}$-stability analysis of uncertain polynomial matrices

Technical Report TR-IDSIA-2017-01

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Foreword

This report is an extended version of the paper *A unified framework for deterministic and probabilistic $\mathcal{D}$-stability analysis of uncertain polynomial matrices* submitted by the authors to the IEEE Transactions on Automatic Control.

Abstract

In control theory, we are often interested in robust $\mathcal{D}$-stability analysis, which aims at verifying if all the eigenvalues of an uncertain matrix lie in a given region $\mathcal{D}$ of the complex plane. Although many algorithms have been developed to provide conditions for an uncertain matrix to be robustly $\mathcal{D}$-stable, the problem of computing the probability of an uncertain matrix to be $\mathcal{D}$-stable is still unexplored. The goal of this paper is to fill this gap by generalizing algorithms for robust $\mathcal{D}$-stability analysis in two directions. First, the only constraint on the stability region $\mathcal{D}$ that we impose is that its complement is a semialgebraic set described by polynomial constraints. This comprises main important cases in robust control theory. Second, the $\mathcal{D}$-stability analysis problem is formulated in a probabilistic framework, by assuming that only few probabilistic information is available on the uncertain parameters, such as support and some moments. We will show how to efficiently compute the minimum probability that the matrix is $\mathcal{D}$-stable by using convex relaxations based on the theory of moments. We will also show that standard robust $\mathcal{D}$-stability is a particular case of the more general probabilistic $\mathcal{D}$-stability problem. Application to robustness and probabilistic analysis of dynamical systems is discussed.

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is available on the uncertain parameters, such as support and some moments. We will show how to efficiently compute the minimum probability that the matrix is $\mathcal{D}$-stable by using convex relaxations based on the theory of moments. We will also show that standard robust $\mathcal{D}$-stability is a particular case of the more general probabilistic $\mathcal{D}$-stability problem. Application to robustness and probabilistic analysis of dynamical systems is discussed.

1 Introduction

1.1 Motivations

Consider a plant described by the transfer function

$$G(s) = \frac{\rho_2}{s + \rho_1},$$

where ρ_1, ρ_2 are uncertain parameters belonging to the intervals $\rho_1 \in [0.035, 0.085]$ and $\rho_2 \in [12, 28]$. Although these parameters can take any value in the corresponding uncertainty intervals, we assume that they are usually close to their nominal values (in this case, the centers of the intervals). The goal is to design a controller that robustly stabilizes the closed-loop system and has a fast unit step response possibly without overshoots. Assume we have designed two controllers $\mathcal{K}_{\text{ROB}}$ and $\mathcal{K}_{\text{PROB}}$ and we want to understand which one is better. First we check the stability requirements. Both of them robustly stabilize the closed-loop system. Then we check the performance by evaluating the unit step response of the controlled system for 100 different parameter realizations (see Fig. 1). It is evident that $\mathcal{K}_{\text{PROB}}$ (red) is preferable in terms of speed (half settling time). However, $\mathcal{K}_{\text{ROB}}$ (blue) has never overshoots, while $\mathcal{K}_{\text{PROB}}$ has one overshoot out of 100 parameter realizations. Should we then choose $\mathcal{K}_{\text{ROB}}$ or $\mathcal{K}_{\text{PROB}}$?

Assume we were able to translate the knowledge that the parameters ρ_1, ρ_2 are usually close to their nominal values in terms of weak probabilistic constraints, such as $\mathbb{E}[\rho_1] = 0.06$, $\mathbb{E}[\rho_2] = 20$ and $\mathbb{E}[(\rho_1 - 0.06)^2] \leq \sigma_1^2 = 0.0025$, $\mathbb{E}[(\rho_2 - 20)^2] \leq \sigma_2^2 = 0.25$ for instance. Moreover assume that, based on this information, we could compute the probability that the step response of the closed-loop system does not have overshoots and that this probability is $p = 1$ with the controller $\mathcal{K}_{\text{ROB}}$ and $p = 0.95$ with $\mathcal{K}_{\text{PROB}}$. Then, if we accept a 5% probability of having an overshoot, we could claim that $\mathcal{K}_{\text{PROB}}$ is preferable. It is like in car racing: pilots know that if they exceed the track limits (overshoot) this causes them to lose time. However, they accept a small probability of exceeding the limits because in this way they can set faster laps. In this paper, we provide a mathematical framework to quantify this probability.

1.2 Contribution

In control theory for *Linear Time-Invariant* (LTI) systems, robust (or probabilistic) stability and performance requirements can be formulated in terms of robust (resp., probabilistic) $\mathcal{D}$-stability analysis, which aims at verifying if (resp., compute the probability that) all the eigenvalues of an uncertain matrix lie in a given region $\mathcal{D}$ of the complex plane. In this paper, we present a unified framework to assess robust and probabilistic $\mathcal{D}$-stability of uncertain matrices. Specifically, the contribution of the paper is twofold:

1. a novel approach for analysing robust $\mathcal{D}$-stability of an uncertain matrix $A(\rho)$ is proposed. The entries of the matrix $A(\rho)$ depend polynomially on an uncertain parameter vector ρ, which is assumed to take values in a closed semialgebraic set Δ described by polynomial constraints. The only assumption on the stability region $\mathcal{D}$ is that its complement is a semialgebraic set (not necessarily convex), described by polynomial constraints in the complex plane. The addressed problem is quite general and it includes, among others, the analysis of robust nonsingularity, Hurwitz or Schur stability of a family of matrices with interval, polytopic or 2-norm bounded perturbations.

2. the $\mathcal{D}$-stability analysis problem is formulated in a probabilistic framework, by assuming that the uncertain parameters ρ are described by a set of non a-priori specified probability measures. Only the support and some moments (e.g., mean and variance) of the probability measures characterizing the uncertainty ρ are assumed to be known. This is an approach to robustness, based on *coherent lower previsions* [1] (also referred to as *Imprecise Probability*) that has been recently developed in filtering theory [2, 3, 4]. Specifically, we seek the "worst-case probabilistic scenario", which requires to compute, among all possible probability measures satisfying the assumptions, the smallest probability of the uncertain matrix $A(\rho)$ to be $\mathcal{D}$-stable.

The latter result allows us to take into account not only the information about the range of the uncertain parameter ρ (i.e., $\rho \in \Delta$), but also information such as: (1) the nominal value of ρ (e.g., the center of the uncertainty set Δ); (2) the variability of ρ w.r.t. its nominal value and so on. This information is not taken into account in standard approaches for $\mathcal{D}$-stability analysis, but it allows us to reduce the conservativeness of the obtained results, at the price of guaranteeing $\mathcal{D}$-stability within a given level of probability. We can for instance determine if a family of matrices is $\mathcal{D}$-stable with probability 0.90 or 0.95 or 0.99, etc..

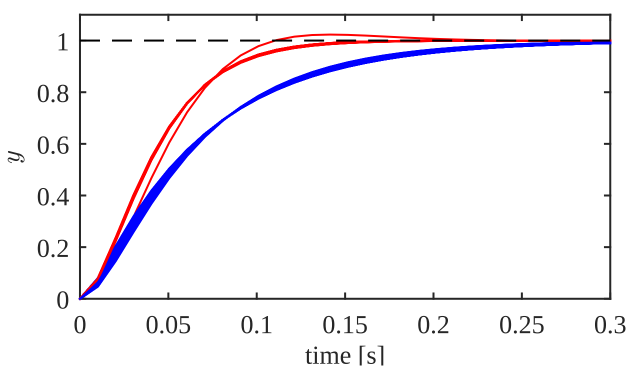

Figure 1: Closed-loop step response with controller $\mathcal{K}_{\text{ROB}}$ (blue) and $\mathcal{K}_{\text{PROB}}$ (red) for 100 different realization of the uncertain parameters.

To this end, we develop a unified framework for deterministic (robust) and probabilistic $\mathcal{D}$-stability analysis. A semi-infinite linear program is formulated and then relaxed, by exploiting the Lasserre's hierarchy [5], into a sequence of (convex) *semidefinite programming* (SDP) problems of finite size.

1.3 Related works

Evaluating the properties of the eigenvalues of a family of matrices (e.g., robust nonsingularity, maximum real part of the eigenvalues or spectral radius) is an NP-hard problem [6, 7, 8], and there is a vast literature addressing this research topic.

Algorithms for checking Hurwitz and Schur stability of symmetric interval matrices are proposed in [9, 10, 11], where it is shown that testing the nonsingularity of symmetric interval matrices requires to calculate a finite number of determinants, and this number grows exponentially in the matrix size, limiting the applicability of these methods to small scale problems. A branch and bound algorithm is then proposed in [12] to solve larger scale problems. Interval and polytopic matrices are considered in [13, 14, 15, 16, 17, 18, 19, 20]. The works in [13, 14] derive intervals where the real eigenvalues of interval matrices are guaranteed to belong to, and a vertex result is presented in [15] to reduce the computational load in evaluating quadratic stability of interval matrices. Results in [15] can be also used in the case of multiaffine interval matrix uncertainty. Bernstein expansion is used in [16] to check robust nonsingularity of a polytope of real matrices, and sufficient LMI conditions coming from the Lyapunov theory are derived in [17, 18, 19, 20] for checking robust Hurwitz and Schur stability of matrices with polytopic uncertainty. In [21, 22, 23, 24], less conservative LMI conditions to check robust $\mathcal{D}$-stability of uncertain polynomial matrices are derived. A method based on the *structured singular value* and on its variant, the *skewed structured singular value*, is proposed in [25] to analyse the spectrum of uncertain matrices expressed in a *linear fractional representation*. Numerically efficient algorithms for computing (lower bounds of) the extreme points (e.g., maximum real part and maximum modulus) of the ε-pseudospectrum of a matrix A are proposed in [26] and [27], where the ε-pseudospectrum of a given matrix A is defined as the set of the eigenvalues of the perturbed matrix $A+E$, for all $\|E\| \leq \varepsilon$. Both the Frobenius and 2-norm are used to measure the "amplitude" of the perturbation E, and structured perturbations can be also handled. Since lower bounds on the maximum real part and on the maximum modulus of the ε-pseudospectrum are computed, necessary conditions for robust Hurwitz and Schur stability of the uncertain matrix $A+E$ can be derived. NP-hard robust matrix analysis problems are tackled in [28, 29, 30] through *randomized algorithms*, which run in polynomial time, at the price of providing an erroneous answer (specifically, a false positive) with some probability. Other contributions addressing robust $\mathcal{D}$-stability analysis, with applications in systems and control theory, can be found in [31, 32, 33, 34, 35, 36, 37, 38] and reference therein.

The list of reviewed works is far from being exhaustive, but it points out the efforts made by researchers in the last decades to develop methodologies that, in many cases, can be applied to tackle specific robust $\mathcal{D}$-stability analysis problems (e.g., robust nonsingularity, Hurwitz or Schur stability) under specific assumptions on the structure of the uncertainty (e.g., interval, polytopic, or 2-norm bounded uncertainty).

In the context of the present paper, it is worth mentioning the works [39] and [40], where two approaches based on Lasserre's hierarchy are proposed to approximate the stability region of univariate polynomials with uncertain coefficients. These results can be also used to assess robust stability of uncertain polynomial matrices. However, unlike the method proposed in this paper, [39] and [40] do not consider the probabilistic scenario and a restricted subset of stability regions $\mathcal{D}$ can be handled (for instance, to the best of the authors' knowledge, the complex plane without the imaginary axis cannot be considered as a stability set). Furthermore, a deep Lasserre's hierarchy may be required in [39] and [40] to achieve non-conservative results (as discussed in the example reported in Section 6.1).

1.4 Paper organization

The paper is organized as follows. The notation used throughout the paper is introduced in Section 2. The $\mathcal{D}$-stability analysis problem is formally defined in Section 3, and a unified framework for deterministic and probabilistic analysis is provided. The main theorems and results are reported in Section 4, where it is shown that the $\mathcal{D}$-stability analysis problem can be formulated as a semi-infinite linear program. Convex relaxation techniques based on the Lasserre's hierarchy [5] and aiming at computing the solution of the formulated semi-infinite linear program are described in Section 5. Applications of the proposed method are discussed in Section 6, along with simulation examples and a comparison with existing approaches for robust $\mathcal{D}$-stability analysis. A simple running example is also used throughout the whole paper for illustrative purposes.

2 Notation

Let us denote with $x_{\rm re}$ and $x_{\rm im}$ the real and imaginary part, respectively, of a complex vector x. Let z_i be the i-th component of a vector $z \in \mathbb{R}^{n_z}$. Let $\mathbb{N}$ be the set of natural numbers and $\mathbb{N}_0^{n_z}$ the set of n_z-dimensional vectors with non-negative integer components.

For a given integer τ, $\mathcal{A}_\tau^{n_z}$ is the set defined as $\{\alpha \in \mathbb{N}_0^{n_z} : \sum_{i=1}^{n_z} \alpha_i \leq \tau\}$. We will use the shorthand notation z^α for $z^\alpha = z_1^{\alpha_1} \cdots z_{n_z}^{\alpha_{n_z}} = \prod_{i=1}^{n_z} z_i^{\alpha_i}$. Let us denote with $\mathbb{R}_\tau[z]$ the set of real-valued polynomials in the variable $z \in \mathbb{R}^{n_z}$ with degree less than or equal to τ, and let $b_\tau(z)$ be the canonical basis of $\mathbb{R}_\tau[z]$, i.e., $b_\tau(z) = \{z^\alpha\}_{\alpha \in \mathcal{A}_\tau^{n_z}}$. Denote with $\{\boldsymbol{g}_\alpha\}_{\alpha \in \mathcal{A}_\tau^{n_z}}$ the coefficients of the polynomial $g \in \mathbb{R}_\tau[z]$ in the canonical basis $b_\tau(z)$, i.e., $g(z) = \sum_{\alpha \in \mathcal{A}_\tau^{n_z}} \boldsymbol{g}_\alpha z^\alpha$. In the case g is an n_g-dimensional vector of polynomials in $\mathbb{R}_\tau[z]$, we denote with $\boldsymbol{g}_{i,\alpha}$ the coefficients of the polynomial g_i in the basis $b_\tau(z)$. Let us denote with $deg(g)$ the degree of the polynomial g.

Let P_z be the cumulative distribution function of a Borel probability measure $\Pr_z$ on $\mathbb{R}^{n_z}$. To understand the relationship between $\Pr_z$ and P_z, we can for instance consider $\mathbb{R}$ and in this case we have that $P_z(z) = \Pr_z(-\infty, z]$ – this definition can easily be extended to $\mathbb{R}^{n_z}$. Because of the equivalence between Borel probability measures and cumulative distributions, hereafter we will use interchangeably $\Pr_z$ and P_z. For an integer $\tau \geq 0$, let $m = \{m_\alpha\}_{\alpha \in \mathcal{A}_\tau^n}$ be the sequence of moments of a probability measure $\Pr_z$ on $\mathbb{R}^{n_z}$, i.e., $m_\alpha = \int z^\alpha dP_z(z)$.

3 Problem setting

3.1 Uncertainty description

Consider an uncertain square real matrix $A(\rho)$ of size n_a, whose entries depend polynomially on an uncertain parameter vector $\rho \in \mathbb{R}^{n_\rho}$. The uncertain vector ρ is assumed to belong to a compact semialgebraic uncertainty set Δ, defined as

$$\Delta = \{\rho \in \mathbb{R}^{n_\rho} : g_i(\rho) \geq 0, \quad i = 1, \ldots, n_g\}, \tag{1}$$

where g_i are real-valued polynomial functions of ρ.

Example 1.1 *Let us introduce a simple example which will be used throughout the paper for illustrative purposes. Let the uncertain matrix be*

$$A(\rho) = \begin{bmatrix} \rho - 1 & 0 \\ 0 & -1 \end{bmatrix}, \tag{2}$$

with $\rho \in \Delta = [0, 1]$*. According to the notation in* (1)*, the set* Δ *is written as:*

$$\Delta = \{\rho \in \mathbb{R} : g_1(\rho) \doteq \rho \geq 0, \quad g_2(\rho) \doteq 1 - \rho \geq 0\}.$$

■

We also assume to have some probabilistic information on the uncertain vector ρ. Specifically, given n_f real-valued polynomial functions f_i $(i = 1, \ldots, n_f)$ called *generalized polynomial moment functions* (gpmfs) and defined on Δ, we assume that the probabilistic information on the vector ρ is represented by the expectations of the gpmfs f_i, i.e.,

$$\mathbb{E}[f_i] = \int_\Delta f_i(\rho) dP_\rho(\rho) = \mu_i, \quad i = 1, \ldots, n_f, \tag{3}$$

where the integral is a Lebesgue-Stieltjes integral with respect to the cumulative distribution function P_ρ of a Borel probability measure Pr_ρ on Δ[1] and $\mu_i \in \mathbb{R}$ are finite and known.[2]

We will always assume that $f_1(\rho) = 1$ and since Pr_ρ is a probability measure it follows that $\mu_1 = 1$. In other words, we have

$$\mathbb{E}[f_1] = \int_\Delta dP_\rho(\rho) = 1,$$

which expresses the fact that Pr_ρ is a probability measure with support on Δ:

$$\mathrm{Pr}_\rho(\rho \in \Delta) = \int_\Delta dP_\rho(\rho) = 1.$$

[1] The sample space is $\mathbb{R}^{n_\rho}$ and we are considering the Borel σ-algebra. Δ is assumed to be an element of the σ-algebra.

[2] Although equality constraints on the gpmfs f_i are considered in (3), the methodology discussed in the paper can also be used in the case of inequality constraints.

Note that the knowledge of the expectation of n_f gpmfs f_i is not enough to uniquely define the measures of probability $\Pr_\rho$, thus we consider the set of all probability measures $\Pr_\rho$ which are compatible with the information in (3):

$$\mathcal{P}_\rho = \left\{ P_\rho : \int_\Delta f_i(\rho) dP_\rho(\rho) = \mu_i, \quad i = 1, \ldots, n_f \right\}. \tag{4}$$

With some abuse of terminology, when in the rest of the paper we state that the probability measures $\Pr_\rho$ belong to $\mathcal{P}_\rho$, we actually mean that the corresponding cumulative distribution functions P_ρ belong to $\mathcal{P}_\rho$.

Example 1.2 *Let us continue the running Example 1.1. We consider two cases.*

1. *In the first case, the probabilistic information about ρ is expressed by the set of probability measures:*

$$\mathcal{P}_\rho^{(1)} = \left\{ P_\rho : \int_0^1 dP_\rho(\rho) = 1 \right\}. \tag{5}$$

This means that only the support $\Delta = [0, 1]$ of the probability measures P_ρ is known.

2. *In the second case, the probabilistic information about ρ is expressed by:*

$$\mathcal{P}_\rho^{(2)} = \left\{ P_\rho : \int_0^1 dP_\rho(\rho) = 1, \quad \int_0^1 f_2(\rho)\, dP_\rho(\rho) = 0.5 \right\}, \tag{6}$$

with $f_2(\rho) = \rho$. This means that both the support and the first moment (i.e., the mean assumed to be 0.5) of the probability measures $\Pr_\rho$ are known. The value of the mean equal to 0.5 can be interpreted as a knowledge on the nominal value of ρ in the interval $[0, 1]$ and, on average, we expect ρ to be equal to 0.5. Note that we may also assume that other moments of ρ are known; for instance we may know the variability of ρ w.r.t. the mean 0.5 (i.e., the variance). This case will be considered in the examples reported in Section 6.

■

The problems reported in the next paragraphs are addressed in this work.

3.2 Probabilistic $\mathcal{D}$-stability analysis

A matrix is $\mathcal{D}$-stable if all the eigenvalues belong to a given region $\mathcal{D}$. In this paper, we assume that the stability region $\mathcal{D}$ is an (open) subset of the complex plane, whose complement $\mathcal{D}^c = \mathbb{C} \setminus \mathcal{D}$ (instability region) is a closed semialgebraic set described by

$$\begin{aligned} \mathcal{D}^c = \{\lambda \in \mathbb{C} : \lambda = \lambda_{\text{re}} + j\lambda_{\text{im}}, \quad \lambda_{\text{re}}, \lambda_{\text{im}} \in \mathbb{R}, \\ d_i(\lambda_{\text{re}}, \lambda_{\text{im}}) \geq 0, i = 1, \ldots, n_d\}, \end{aligned} \tag{7}$$

with d_i being real-valued polynomials in the real variables λ_{re} and λ_{im}. Note that $\mathcal{D}$ can be, for instance, the open left half plane, the unitary disk centered in the origin, or the complex plane without the imaginary axis. Therefore, this assumption cover all important cases in stability analysis.

Among all the probability measures belonging to $\mathcal{P}_\rho$, we want to find the "worst-case scenario" given by the measure of probability $\Pr_\rho$ which provides the lower probability that $A(\rho)$ has all the eigenvalues in $\mathcal{D}$ or, equivalently, the upper probability $\overline{p} = 1 - \underline{p}$ that $A(\rho)$ has at least an eigenvalue in $\mathcal{D}^c$. In this way, we can claim that the probability of the matrix $A(\rho)$ to be $\mathcal{D}$-stable w.r.t. the uncertainties ρ is greater than or equal to $\underline{p}$ (equiv. $1 - \overline{p}$).

Formally, we are interested in solving the following eigenvalue location problem.

Problem 1 ***[Probabilistic eigenvalue violation]*** *Given the uncertain matrix $A(\rho)$, the uncertain parameter vector ρ with (unknown) measure of probability $\Pr_\rho$ belonging to $\mathcal{P}_\rho$, and a stability region $\mathcal{D}$, compute*

$$\overline{p} = \sup_{P_\rho \in \mathcal{P}_\rho} \Pr_\rho \left(\Lambda(A(\rho)) \nsubseteq \mathcal{D} \right), \tag{8}$$

where $\Lambda(A(\rho))$ is the spectrum of the matrix $A(\rho)$, or equivalently,

$$\overline{p} = \sup_{P_\rho \in \mathcal{P}_\rho} \Pr_\rho \left(\lambda_i(A(\rho)) \in \mathcal{D}^c \right), \text{ for some } i = 1, \ldots, n_a. \tag{9}$$

■

Example 1.3 *Let us again consider the running example. As a stability region, we consider the open left half-plane*

$$\mathcal{D} = \{\lambda \in \mathbb{C} \mid \lambda_{re} < 0\},$$

whose complement is the semi-algebraic set:

$$\mathcal{D}^c = \{\lambda \in \mathbb{C} \mid d_1(\lambda_{re}) \doteq \lambda_{re} \geq 0\}.$$

Since the eigenvalues of the matrix $A(\rho)$ in (2) are -1 and $\rho - 1$, the only eigenvalue that can lead to instability is $\rho - 1$. Therefore, in this case the problem (9) becomes:

$$\overline{p} = \sup_{P_\rho \in \mathcal{P}_\rho} \Pr_\rho \left(\rho - 1 \geq 0 \right), \tag{10}$$

where we have exploited the fact that $\mathcal{D}^c = \{\lambda \in \mathbb{C} \mid \lambda_{re} \geq 0\}$ and $\lambda_{re} = \rho - 1$. Thus, problem (9) aims at computing the upper probability that the matrix $A(\rho)$ is not $\mathcal{D}$-stable, given the probabilistic information on ρ expressed by the set of feasible cumulative distribution functions $\mathcal{P}_\rho$. ■

The following theorem shows that the challenging problem of verifying deterministic (robust) $\mathcal{D}$-stability of $A(\rho)$ is a special case of Problem 1.

Theorem 1 (Deterministic eigenvalue violation) *In the case the only information on ρ is the support Δ of the probability measures $\Pr_\rho$ (namely, we only know that $\rho \in \Delta$), the solution $\overline{p}$ of problem (8) can be either 1 or 0. Specifically, $\overline{p} = 1$ if $A(\rho)$ is not robustly $\mathcal{D}$-stable w.r.t. the uncertainty set Δ, $\overline{p} = 0$ otherwise.*

Proof irst of all observe that

$$\Pr_\rho\left(\Lambda(A(\rho)) \nsubseteq \mathcal{D}\right) = \int_\Delta (1 - \mathbb{I}_\mathcal{D}\left(\Lambda(A(\rho))\right))\, dP_\rho(\rho),$$

where $1 - \mathbb{I}_\mathcal{D}\left(\Lambda(A(\rho))\right)$ is the complement of the indicator function:

$$\mathbb{I}_\mathcal{D}(\Lambda(A(\rho))) = \begin{cases} 1 & \text{if } \Lambda(A(\rho)) \subseteq \mathcal{D}, \\ 0 & \text{otherwise,} \end{cases} \tag{11}$$

and $P_\rho \in \mathcal{P}_\rho$ with

$$\mathcal{P}_\rho = \left\{ P_\rho : \int_\Delta dP_\rho(\rho) = 1 \right\}. \tag{12}$$

$\mathcal{P}_\rho$ includes all the probability measures supported by Δ and so it also includes atomic measures (Dirac's delta) with support in Δ. Hence, assume that the matrix $A(\rho)$ is not robustly $\mathcal{D}$-stable against Δ. Thus, there exists $\hat{\rho} \in \Delta$ such that $\Lambda(A(\hat{\rho})) \nsubseteq \mathcal{D}$. Then we can take $\Pr_\rho$ equal to the Dirac's delta centred on $\hat{\rho}$ and we have that

$$\Pr_\rho\left(\Lambda(A(\hat{\rho})) \nsubseteq \mathcal{D}\right) = 1.$$

Similarly, assume that the matrix $A(\rho)$ is $\mathcal{D}$-stable for any $\rho \in \Delta$. Then $(1 - \mathbb{I}_\mathcal{D}\left(\Lambda(A(\rho))\right)) = 0$ for any $\rho \in \Delta$. Thus, $\Pr_\rho\left(\Lambda(A(\hat{\rho})) \nsubseteq \mathcal{D}\right) = 0$.

Example 1.4 *Let us go back to our running example assuming the set of probability measures (13). Theorem 1 proves that the robust $\mathcal{D}$-stability analysis problem can be reformulated in a probabilistic way by writing the deterministic constraint $\rho \in \Delta = [0,1]$ as the equivalent probabilistic constraint:*

$$P_\rho \in \mathcal{P}_\rho^{(1)} = \left\{ P_\rho : \int_0^1 dP_\rho(\rho) = 1 \right\}. \tag{13}$$

We can then determine the upper probability that the matrix is unstable by solving the optimization problem:

$$\overline{p} = \sup_{P_\rho \in \mathcal{P}_\rho^{(1)}} \Pr_\rho\left(\rho - 1 \geq 0\right). \tag{14}$$

The solution of the above optimization problem is given by the probability measure $\Pr_\rho = \delta_{(1)}$, i.e., an atomic measure (Dirac's delta) centered at $\rho = 1$. In fact, this measure belongs to $\mathcal{P}_\rho^{(1)}$ since

$$\int_0^1 \delta_{(1)}(\rho) d\rho = 1,$$

and therefore is compatible with the probabilistic information on ρ. Moreover, for this measure, we have:

$$\Pr_\rho\left(\rho - 1 \geq 0\right) = \int_0^1 \mathbb{I}_{[1,\infty)}(\rho)\delta_{(1)}(\rho) d\rho = 1,$$

where $\mathbb{I}_{[1,\infty)}(\rho)$ is the indicator function of the set $[1,\infty)$. Since $\overline{p} = 1$, we can conclude that there exists at least one value of ρ in Δ such that the matrix is not $\mathcal{D}$-stable. ■

Theorem 1 shows that the deterministic $\mathcal{D}$-stability analysis problem is a particular case of probabilistic $\mathcal{D}$-stability analysis. Although the result in Theorem 1 is quite intuitive, it is fundamental to formulate, in a rigorous way, the deterministic and the probabilistic $\mathcal{D}$-stability analysis problem in a unified framework. In fact, one could erroneously think that the probabilistic constraint equivalent to $\rho \in \Delta$ is

$$\int_{\Delta} \frac{1}{|\Delta|} d\rho = 1, \tag{15}$$

where $|\Delta|$ is the Lebesgue measure of Δ, i.e., $\Pr_\rho$ is equal to the uniform distribution on Δ. This is not the case as illustrated in the following example.

Example 1.5 *If in the running example we translate the (deterministic) information $\rho \in \Delta$ as in (15) (with $|\Delta| = 1$), the probability that the matrix is unstable would be equal to zero, since the only value that gives instability ($\rho = 1$) has zero Lebesgue measure. The mistake here is that the uniform distribution is just one of the possible probability measures with support on Δ. There are infinite of such distributions and, as discussed above, the one that gives rise to instability is an atomic measure on the value $\rho = 1$. Thus, the equivalent of the constraint $\rho \in \Delta$ is (13) and not (15).* ■

4 A moment problem for $\mathcal{D}$-stability analysis

As shown in Theorem 1, the problem of evaluating (deterministic) robust $\mathcal{D}$-stability of an uncertain matrix $A(\rho)$ is a particular case of probabilistic $\mathcal{D}$-stability analysis. However, for the sake of exposition, we first provide results in the deterministic setting, where only the set Δ where the uncertainty ρ belongs to is assumed to be known. The probabilistic scenario, where the expectations of the generalized polynomial moment functions f_i of ρ are known (eq. (3)), will be discussed later.

4.1 Checking determinist $\mathcal{D}$-stability

The following theorem (based on a proper extension of the results recently proposed by one of the authors in [41] to compute the *structured singular value* of a matrix) provides necessary and sufficient conditions to check determinist (robust) $\mathcal{D}$-stability of the matrix $A(\rho)$ against the uncertainty set Δ.

Theorem 2 *All eigenvalues of the matrix $A(\rho)$ are located in the set $\mathcal{D}$ for all uncertainties $\rho \in \Delta$ if and only if the solution of the following (nonconvex) optimization problem is 0:*

$$\max_{x \in \mathbb{C}^{n_a}, \rho \in \Delta, \lambda \in \mathbb{C}} \quad \|x\|_2^2 \tag{16a}$$

$$s.t. \quad (A(\rho) - \lambda I)\, x = 0, \quad \|x\|^2 \leq 1, \quad \lambda \in \mathcal{D}^c. \tag{16b}$$

Proof First, the "only if" part is proven. If all the eigenvalues of $A(\rho)$ are located in the set $\mathcal{D}$ (or equivalently, no eigenvalue of $A(\rho)$ belongs to the complement set $\mathcal{D}^c$), there exists no value $\lambda \in \mathcal{D}^c$ and $\rho \in \Delta$ which make the matrix $A(\rho) - \lambda I$ singular. Thus, only the trivial solution $x = 0$ satisfies the constraint $(A(\rho) - \lambda I)\, x = 0$. Therefore, the solution of problem (16) is equal to zero.

The "if" part is proven by contradiction. Assume there exists an uncertainty $\rho \in \Delta$ such that an eigenvalue λ_i of $A(\rho)$ belongs to $\mathcal{D}^c$. Thus, the corresponding eigenvector $x^* \neq 0$ satisfies the constraint $(A(\rho) - \lambda_i I)\, x^* = 0$. Furthermore, for any $\beta \in \mathbb{C}$, also $x = \beta x^*$ satisfies the constraint $(A(\rho) - \lambda_i I)\, x = 0$. Thus, the supremum of the 2 norm of the set of vectors x satisfying $(A(\rho) - \lambda_i I)\, x = 0$ is infinity. Since the constraint $\|x\|^2 \leq 1$ is present in (16), the solution of problem (16) is 1, contradicting the hypothesis.

Corollary 1 *There exists an uncertainty $\rho \in \Delta$ such that at least an eigenvalue λ_i of $A(\rho)$ does not belong to $\mathcal{D}$ if and only if the solution of the problem* (16) *is* 1*.*

Proof It follows straightforwardly from Theorem 2 and its proof.

Example 1.6 *In the explanatory example considered so far, problem* (16) *is:*

$$\max_{x \in \mathbb{R}^2,\, \rho \in [0 \;\; 1],\, \lambda_{\text{re}} \in \mathbb{R}} \quad \|x\|_2^2 \tag{17a}$$

s.t.

$$\begin{bmatrix} \rho - 1 - \lambda_{\text{re}} & 0 \\ 0 & -1 - \lambda_{\text{re}} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \;\; \|x\|^2 \leq 1, \;\; \lambda_{\text{re}} \geq 0. \tag{17b}$$

where we have exploited the fact that, since $A(\rho)$ is a real symmetric matrix, its eigenvalues are real. A feasible point of problem (17) *is $\rho = 1$, $\lambda_{\text{re}} = 0$, and $[x_1 \;\; x_2]^\top = [1 \;\; 0]^\top$. At this point, $\|x\|^2 = 1$, which is the maximum of $\|x\|_2^2$ under the constraint $\|x\|_2^2 \leq 1$. Thus, according to Theorem 2 and Corollary 1, the matrix $A(\rho)$ is not robustly $\mathcal{D}$-stable.* ■

4.2 Checking probabilistic $\mathcal{D}$-stability

Let us now focus on the probabilistic $\mathcal{D}$-stability analysis problem, which aims at computing $\overline{p}$, namely, the upper probability among the probability measures in $\mathcal{P}_\rho(\mu)$ of the matrix $A(\rho)$ to have at least an eigenvalue in the instability region $\mathcal{D}^c$ (see Problem 1). The following theorem, which can be seen as the probabilistic version of Theorem 2 and Corollary 1, shows how the computation $\overline{p}$ can be formulated as a moment optimization problem.

Theorem 3 *Given the uncertain matrix $A(\rho)$, the uncertain parameter vector ρ whose measures of probability $\Pr_\rho(\rho)$ are constraint to belong to $\mathcal{P}_\rho$, and the instability region $\mathcal{D}^c$, the upper probability $\overline{p}$ (defined in* (9)*) of the matrix $A(\rho)$ to have at least an eigenvalue in $\mathcal{D}^c$ is given by the solution of the following optimization problem:*

$$\overline{p} = \sup_{P_{\rho,x,\lambda}} \iiint \|x\|^2 dP_{\rho,x,\lambda}(\rho, x, \lambda) \tag{18a}$$

$s.t.$

$$\iiint dP_{\rho,x,\lambda}(\rho, x, \lambda) = 1, \tag{18b}$$

$$\int_{\rho\in\Delta} \int_{\|x\|^2\leq 1} \int_{\lambda\in\mathcal{D}^c} dP_{\rho,x,\lambda}(\rho, x, \lambda) = 1, \tag{18c}$$

$$\int_{\rho\in\Delta} \int_{\|x\|^2\leq 1} \int_{\lambda\in\mathcal{D}^c} f_i(\rho) dP_{\rho,x,\lambda}(\rho, x, \lambda) = \mu_i, \ i=2,\ldots,n_f, \tag{18d}$$

$$\iiint_{(A(\rho)-\lambda I)x=0} dP_{\rho,x,\lambda}(\rho, x, \lambda) = 1, \tag{18e}$$

with $P_{\rho,x,\lambda}$ being the joint cumulative distribution function of the variables (ρ, x, λ).

Observe that (18c) is just the moment constraint:

$$\int_{\rho\in\Delta} \int_{\|x\|^2\leq 1} \int_{\lambda\in\mathcal{D}^c} f_1(\rho) dP_{\rho,x,\lambda}(\rho, x, \lambda) = \mu_1,$$

which has been explicited to highlight the support of $P_{\rho,x,\lambda}$.

Proof irst, note that the constraints (18b) and (18c) guarantee that $P_{\rho,x,\lambda}$ is a cumulative distribution function of a probability distribution $\Pr_{\rho,x,\lambda}$, whose marginals $\Pr_\rho$, $\Pr_x$ and $\Pr_\lambda$ are supported by Δ, $\{x \in \mathbb{C}^{n_a} : \|x\|^2 \leq 1\}$, and $\mathcal{D}^c$, respectively. Furthermore, the constraint in (18d) guarantees that $P_\rho \in \mathcal{P}_\rho$, in fact:

$$\begin{aligned} &\int_{\rho\in\Delta} \int_{\|x\|^2\leq 1} \int_{\lambda\in\mathcal{D}^c} f_i(\rho) dP_{\rho,x,\lambda}(\rho, x, \lambda) \\ &= \int_{\Delta} f_i(\rho) dP_\rho(\rho) = \mu_i, \ i=2,\ldots,n_f. \end{aligned} \tag{19}$$

Let us now consider the constraint (18e). The following two situations may occur:

1. the pair $\hat{\rho}$ and $\hat{\lambda}$ does not make the matrix $A(\hat{\rho}) - \hat{\lambda} I$ singular (namely, $\hat{\lambda}$ is not an eigenvalue of $A(\hat{\rho})$). Then, the only value of x in the integral domain $(A(\rho) - \lambda I)\, x = 0$ is $x = 0$. Thus, only a joint cumulative probability distribution $P_{\rho,x,\lambda}$ with marginal probability distribution $\Pr_x = \delta_{(0)}(x)$ satisfies (18e).

2. the pair $\hat{\rho}$ and $\hat{\lambda}$ makes the matrix $A(\hat{\rho}) - \hat{\lambda} I$ singular (namely, $\hat{\lambda}$ is an eigenvalue of $A(\hat{\rho})$). Thus, any left eigenvector $\hat{x} \neq 0$ of the matrix $A(\hat{\rho})$ associated to the eigenvalue $\hat{\lambda}$ satisfies $(A(\rho) - \lambda I)\, x = 0$. Thus, the marginal dP_x of the joint $dP_{\rho,x,\lambda}$ is not constraint to have its mass centered in $x = 0$. It depends on the value of ρ, λ, i.e., $P_x(\cdot|\rho, \lambda)$, we can the decompose $dP_{\rho,x,\lambda}$ as $dP_{\rho,x,\lambda} = dP_x(\cdot|\rho, \lambda) dP_{\rho,\lambda}$.

Based on the considerations above, the support $S_x(\cdot|\rho, \lambda)$ of the marginal probability distribution $\Pr_x(\cdot|\rho, \lambda)$ is either

$$S_x(\cdot|\rho, \lambda) = \{0\} \quad \text{if } A(\rho) - \lambda I \text{ is nonsingular,} \tag{20}$$

or

$$S_x(\cdot|\rho, \lambda) = \left\{x : \|x\|^2 \leq 1,\ (A(\rho) - \lambda I)\, x = 0\right\} \quad \text{if } A(\rho) - \lambda I \text{ is singular.} \tag{21}$$

Let us rewrite the joint $dP_{\rho,x,\lambda}$ as $dP_{\rho,x,\lambda} = dP_x(\cdot|\rho, \lambda) dP_{\rho,\lambda}$ and let us split the objective function in (18a) as:

$$\iiint \|x\|^2 dP_{\rho,x,\lambda}(\rho, x, \lambda) \tag{22a}$$

$$= \iiint_{\substack{A(\rho)-\lambda I \\ \text{nonsingular}}} \|x\|^2 dP_x(x|\rho, \lambda) dP_{\rho,\lambda}(\rho, \lambda) + \tag{22b}$$

$$+ \iiint_{\substack{A(\rho)-\lambda I \\ \text{singular}}} \|x\|^2 dP_x(x|\rho, \lambda) dP_{\rho,\lambda}(\rho, \lambda). \tag{22c}$$

Let us consider the term (22b). Based on the above considerations, for any probability measure satisfying the constraints (18c)-(18e), we have:

$$\iiint_{\substack{A(\rho)-\lambda I \\ \text{nonsingular}}} \|x\|^2 dP_x(x|\rho, \lambda) dP_{\rho,\lambda}(\rho, \lambda) \tag{23a}$$

$$= \iiint \|x\|^2 \delta_{(0)}(x) dx dP_{\rho,\lambda}(\rho, \lambda) = 0. \tag{23b}$$

Let us consider the term (22c). For any probability measure satisfying the constraints (18c)-(18e), we have:

$$\iiint_{\substack{A(\rho)-\lambda I \\ \text{singular}}} \|x\|^2 dP_x(x|\rho, \lambda) dP_{\rho,\lambda}(\rho, \lambda) \tag{24a}$$

$$\leq \iiint_{\substack{A(\rho)-\lambda I \\ \text{singular}}} dP_{\rho,\lambda}(\rho, \lambda) = \Pr_\rho\left(\Lambda\left(A(\rho)\right) \not\subseteq \mathcal{D}\right) = \overline{p}, \tag{24b}$$

where the inequality comes from the fact that the support of $\Pr_x(x|\rho, \lambda)$ is bounded by $\|x\|^2 \leq 1$ (see eq. (21)). Among all the feasible conditional distributions $\Pr_x(\cdot|\rho, \lambda)$, which are constrained to have support S_x in (21), let us consider the Dirac's function $\delta_{(\hat{x})}$ centered at $\hat{x}$, with $\hat{x} : \|\hat{x}\|^2 = 1$. For such a distribution, the term (24a) is equal to:

$$
\begin{aligned}
&\iiint\limits_{\substack{A(\rho)-\lambda I\\ \text{singular}}} \|x\|^2 dP_x(x|\rho,\lambda)dP_{\rho,\lambda}(\rho,\lambda)\\
&= \iiint\limits_{\substack{A(\rho)-\lambda I\\ \text{singular}}} \|x\|^2 \delta_{(\hat{x})}(x)dx dP_{\rho,\lambda}(\rho,\lambda)\\
&= \iiint\limits_{\substack{A(\rho)-\lambda I\\ \text{singular}}} dP_{\rho,\lambda}(\rho,\lambda) = \mathrm{Pr}_\rho\left(\Lambda\left(A(\rho)\right) \nsubseteq \mathcal{D}\right) = \overline{p}. \qquad (25)
\end{aligned}
$$

Thus, from (25) and the upper bound in (24b), we have that, at the optimum,

$$
\iiint\limits_{\substack{A(\rho)-\lambda I\\ \text{singular}}} \|x\|^2 dP_x(x|\rho,\lambda)dP_{\rho,\lambda}(\rho,\lambda) = \mathrm{Pr}_\rho\left(\Lambda\left(A(\rho)\right) \nsubseteq \mathcal{D}\right) = \overline{p}. \qquad (26)
$$

By combining eq. (22) with the conditions (23) and (26), the theorem follows.

The intuitive explanation behind the formulation of problem (18) is the following. According to Theorem 2 and Corollary 1, when the optimum of the deterministic problem (16) is achieved, $\|x\|^2 = 1$ if $\Lambda\left(A(\rho)\right) \nsubseteq \mathcal{D}$, 0 otherwise. Thus, when the information on ρ is modeled in terms of probability measures, $\|x\|^2$ becomes a uncertain variable which takes the values:

$$
\|x\|^2 = \begin{cases} 0 & \text{if } \Lambda\left(A(\rho)\right) \subseteq \mathcal{D}, \\ 1 & \text{if } \Lambda\left(A(\rho)\right) \nsubseteq \mathcal{D}. \end{cases}
$$

Thus, the expected value of $\|x\|^2$ (namely, the objective function in (18)) coincides with $\mathrm{Pr}_x(\|x\|^2 = 1)$, which in turn provides $\mathrm{Pr}_\rho\left(\Lambda\left(A(\rho)\right) \nsubseteq \mathcal{D}\right)$.

The constraints in (18c) and (18e) are simply the "probabilistic version" of the determinist constraints in (16), and they are used to describe the support of the probability measures $\mathrm{Pr}_{\rho,x,\lambda}$. The constraint (18d) includes the information in (3) on the (generalized) moments of the probability measures Pr_ρ, i.e., $P_\rho \in \mathcal{P}_\rho$.

Example 1.7 *Let us continue the explanatory example, and consider the case where the probabilistic information on* ρ *is expressed by the set* $\mathcal{P}_\rho^{(2)}$ *(eq. (6)).*

Then, problem (18) *is given by:*

$$\overline{p} = \sup_{P_{\rho,x,\lambda}} \iiint \|x\|^2 dP_{\rho,x,\lambda}(\rho,x,\lambda) \tag{27a}$$

s.t.

$$\int dP_{\rho,x,\lambda}(\rho,x,\lambda) = 1, \tag{27b}$$

$$\int_{\rho\in[0\ 1]} \int_{\|x\|^2\leq 1} \int_{\lambda_{\rm re}\geq 0} dP_{\rho,x,\lambda}(\rho,x,\lambda) = 1, \tag{27c}$$

$$\int_{\rho\in[0\ 1]} \int_{\|x\|^2\leq 1} \int_{\lambda_{\rm re}\geq 0} \rho dP_{\rho,x,\lambda}(\rho,x,\lambda) = 0.5, \tag{27d}$$

$$\iiint_{(A(\rho)-\lambda_{\rm re}I)x=0} dP_{\rho,x,\lambda}(\rho,x,\lambda) = 1. \tag{27e}$$

Because of the constraint (27c)*, the joint distribution* $\mathrm{Pr}_{\rho,x,\lambda}$ *is supported by*

$$\left\{(\rho,x,\lambda_{\rm re}): \quad \rho\in[0\ 1],\ \|x\|^2\leq 1,\ \lambda_{\rm re}\geq 0\right\}.$$

We remind that $A(\rho)$ *is unstable if and only if* $\rho = 1$*. For this value of* ρ*,* $A(\rho)$ *has an eigenvalue in zero. Let us rewrite* $P_{\rho,x,\lambda}$ *as* $P_x(\cdot|\rho,\lambda_{\rm re})P_{\rho,\lambda_{\rm re}}$*. Then, because of* (27e)*, the conditional marginal distribution* $P_x(\cdot|\rho,\lambda_{\rm re})$ *is supported by:*

$$\begin{array}{r}\left\{x\!:\|x_1\|^2\leq 1,\ x_2=0\right\} \text{ if } \rho=1 \text{ and } \lambda_{\rm re}=0,\\ \{0\} \text{ if } \rho\neq 1 \text{ or } \lambda_{\rm re}\neq 0.\end{array}$$

Thus, at the optimum, the objective function of problem (27) *is given by*

$$\int_{\rho=1}\int_{\lambda_{\rm re}=0} dP_{\rho,\lambda}(\rho,\lambda). \tag{28}$$

Among all the probability measures $\mathrm{Pr}_{\rho,\lambda}$ *satisfying the moment constraint* (27d) *on the marginal distribution* Pr_{ρ} *and the constraints* (27c)-(27e)*, the one maximizing* (28) *is given by*

$$\mathrm{Pr}_{\rho,\lambda}(\rho,\lambda) = \left(0.5\delta_{(0)}(\rho)+0.5\delta_{(1)}(\rho)\right)\delta_{(0)}(\lambda_{\rm re}). \tag{29}$$

Thus, the maximum value of the objective function in (28) *is given by:*

$$\begin{array}{l}\displaystyle\int_{\rho=1}\int_{\lambda_{\rm re}=0} dP_{\rho,\lambda}(\rho,\lambda) = \\ \displaystyle\int_{\rho=1}\left(0.5\delta_{(0)}(\rho)+0.5\delta_{(1)}(\rho)\right)d\rho\int_{\lambda_{\rm re}=0}\delta_{(0)}(\lambda_{\rm re})d\lambda_{\rm re} = 0.5.\end{array}$$

Therefore, by exploiting the information on the mean we can reduce the upper probability of instability from 1 *to* 0.5. ■

5 Solving moment problems through SDP relaxations

Note that, in problem (18): (i) the decision variables are the amount of nonnegative mass $\mathrm{Pr}_{\rho,x,\lambda}$ assigned to each point (ρ,x,λ), (ii) the objective function and the constraints are linear in the optimization variables $P_{\rho,x,\lambda}$. Therefore, (18) is a *semi-infinite linear program*, with a finite number of constraints but with infinite number of decision variables. In this section, we show how to use results from the theory-of-moments relaxation proposed by Lasserre in [5], and concerning the characterization of those sequences that are sequence of moments of some probability measures, to relax the semi-infinite linear programming problem (18) into a hierarchy of *semidefinite programming* (SDP) problems of finite dimension.

Let us first introduce the augmented variable vector $z = \begin{bmatrix} x_{\text{re}}^\top & x_{\text{im}}^\top & \rho^\top & \lambda_{\text{re}} & \lambda_{\text{im}} \end{bmatrix}^\top \in \mathbb{R}^{n_z}$ (with $n_z = 2n_a + n_\rho + 2$) and, with some abuse of notation, let us define $h(z) = \|x\|^2$ and $\tilde{f}(z) = f(\rho)$. Problem (18) can be then rewritten in terms of the augmented variable z and the cumulative distribution function P_z as

$$\overline{p} = \sup_{P_z} \int h(z) dP_z(z) \tag{30a}$$

$$s.t.$$

$$\int dP_z(z) = 1, \tag{30b}$$

$$\int \tilde{f}_i(z) dP_z(z) = \mu_i, \; i = 2, \ldots, n_f, \tag{30c}$$

$$\int_{\boldsymbol{Z}} dP_z(z) = 1, \tag{30d}$$

where $\boldsymbol{Z}$ defines the support of the probability measure Pr_z. Thus, based on the definition of the sets Δ (eq. (1)) and $\mathcal{D}^c$ (eq. (7)), the set $\boldsymbol{Z}$ is described by:

$$\begin{aligned} \boldsymbol{Z} = \Big\{ & z = \begin{bmatrix} x_{\text{re}}^\top & x_{\text{im}}^\top & \rho^\top & \lambda_{\text{re}} & \lambda_{\text{im}} \end{bmatrix}^\top : \\ & g_i(\rho) \geq 0, \quad i = 1, \ldots, n_g, \\ & d_i(\lambda_{\text{re}}, \lambda_{\text{im}}) \geq 0, \quad i = 1, \ldots, n_d, \\ & (A(\rho) - \lambda_{\text{re}} I)\, x_{\text{re}} + \lambda_{\text{im}} x_{\text{im}} = 0, \\ & (A(\rho) - \lambda_{\text{re}} I)\, x_{\text{im}} - \lambda_{\text{im}} x_{\text{re}} = 0, \\ & \|x_{\text{re}}\|^2 + \|x_{\text{im}}\|^2 \leq 1 \; \Big\}. \end{aligned} \tag{31}$$

In order to compact the notation, we will rewrite the set $\boldsymbol{Z}$ as:

$$\boldsymbol{Z} = \{ z \in \mathbb{R}^{n_z} : \, q_j(z) \geq 0, \quad j = 1, \ldots, n_q \}, \tag{32}$$

with $q_j(z)$ being real-valued polynomial functions in z, properly defined based on the description of $\boldsymbol{Z}$ in (31).

Example 1.8 *Since in the explanatory example considered so far $A(\rho)$ is a real symmetric matrix, its eigenvalues are real, and thus we considered an augmented variable vector z:*

$$z = [\rho \; \lambda_{re} \; x_1 \; x_2]^\top \in \mathbb{R}^4. \tag{33}$$

The objective function $h(z)$ is $h(z) = z_3^2 + z_4^2$, and the components of the vector-valued function $\tilde{f}(z)$ defining the constraints on the moments is $\tilde{f}_1(z) = 1$ and

$\tilde{f}_2(z) = z_1$. *According to the description in* (32)*, the set* $\boldsymbol{Z}$ *defining the support of the probability measure* Pr_z *is given by:*

$$\begin{aligned}\boldsymbol{Z} = \big\{ & z = [\rho\ \lambda_{re}\ x_1\ x_2]^\top : \\ & q_1(z) \doteq z_1 \geq 0, \quad q_2(z) \doteq 1 - z_1 \geq 0, \\ & q_3(z) \doteq z_2 \geq 0, \\ & q_4(z) \doteq (z_1 - 1)z_3 \geq 0,\ q_5(z) \doteq -(z_1 - 1)z_3 \geq 0, \\ & q_6(z) \doteq z_4 \geq 0,\ q_7(z) \doteq -z_4 \geq 0, \\ & q_8(z) \doteq 1 - z_3^2 - z_4^2 \geq 0 \ \big\}.\end{aligned}$$

■

For an integer $\tau \in \mathbb{N} :\ \tau \geq \tilde{\tau}$, with

$$\tilde{\tau} = \max\left\{1, \max_{i=1,\ldots,n_{\tilde{f}}} \left\lceil \frac{deg(\tilde{f}_i)}{2} \right\rceil, \max_{j=1,\ldots,n_q} \left\lceil \frac{deg(q_j)}{2} \right\rceil \right\}, \tag{34}$$

let us rewrite $h(z) \in \mathbb{R}_{2\tau}[z]$ and each component $\tilde{f}_i(z) \in \mathbb{R}_{2\tau}[z]$ of the vector-valued function $\tilde{f}(z)$ as

$$h(z) = \sum_{\alpha \in \mathcal{A}_{2\tau}^{n_z}} \boldsymbol{h}_\alpha z^\alpha, \qquad \tilde{f}_i(z) = \sum_{\alpha \in \mathcal{A}_{2\tau}^{n_z}} \tilde{\boldsymbol{f}}_{i,\alpha} z^\alpha, \tag{35}$$

where, according to the notation introduced in Section 2, $\boldsymbol{h}_\alpha$ (resp. $\tilde{\boldsymbol{f}}_{i,\alpha}$) are the coefficients of the polynomial $h(z)$ (resp. $\tilde{f}_i(z)$). Based on eq. (35), we can write

$$\int h(z) dP_z(z) = \int \left(\sum_{\alpha \in \mathcal{A}_{2\tau}^{n_z}} \boldsymbol{h}_\alpha z^\alpha \right) dP_z(z) = \sum_{\alpha \in \mathcal{A}_{2\tau}^{n_z}} \boldsymbol{h}_\alpha m_\alpha,$$

where m_α are the moments of the probability measure Pr_z, i.e.,

$$m_\alpha = \int z^\alpha dP_z(z),$$

as introduced in Section 2. Similar considerations hold for the polynomial $\tilde{\boldsymbol{f}}_i(z)$.

Thus, solving problem (30) is equivalent to solve:

$$\begin{aligned}\overline{p} = & \sup_{m=\{m_\alpha\}_{\alpha \in \mathcal{A}_{2\tau}^{n_z}}} \sum_{\alpha \in \mathcal{A}_{2\tau}^{n_z}} \boldsymbol{h}_\alpha m_\alpha \\ & s.t. \\ & \quad \sum_{\alpha \in \mathcal{A}_{2\tau}^{n_z}} \tilde{\boldsymbol{f}}_{i,\alpha} m_\alpha = \mu_i, \quad i = 2, \ldots, n_f, \\ & \quad m \text{ is a sequence of moments generated by a} \\ & \quad\quad \text{probability measure with support on } \boldsymbol{Z}.\end{aligned} \tag{36}$$

Comparing (30) and (36) is evident that now the optimization variables are the moments m_α (real numbers), where the constraint "Pr_z is a probability measure on $\boldsymbol{Z}$" has been replaced by "m is a sequence of moments generated by a probability measure with support on $\boldsymbol{Z}$".

From a straightforward application of Lasserre's hierarchy (see [5] and [42, Sec. 4.1.5]), necessary conditions for the sequence $m = \{m_\alpha\}_{\alpha \in \mathcal{A}_{2\tau}^{n_z}}$ to be a sequence of moments generated by a probability measure $\Pr_z(z)$ with support on $\boldsymbol{Z}$ can be derived. Before discussing the application Lasserre's hierarchy to problem (30), let us introduce the following notation.

For a generic polynomial function $g \in \mathbb{R}_\tau[z]$, let us define the map $L_m(g)$ as:

$$g \mapsto L_m(g) = \int g(z) dP_z(z) = \sum_{\alpha \in \mathcal{A}_\tau^{n_z}} \boldsymbol{g}_\alpha \int z^\alpha dP_z(z) = \sum_{\alpha \in \mathcal{A}_\tau^{n_z}} \boldsymbol{g}_\alpha m_\alpha.$$

Let us define the so-called *moment matrix* $M_\tau(m)$ truncated to order τ as

$$M_\tau(m) = \int b_\tau(z) b_\tau^\top(z) dP_z(z) = L_m(b_\tau(z) b_\tau^\top(z)), \tag{37}$$

with $b_\tau(z)$ defined in Section 2 and where the operator L_m is applied entry-wise to the matrix $b_\tau(z) b_\tau^\top(z)$.

Let us also define the so-called truncated *localizing matrix* $M_\tau(gm)$ of order τ associated with the polynomial g as:

$$M_\tau(gm) = \int g(z) b_\tau(z) b_\tau^\top(z) dP_z(z) = L_m(g(z) b_\tau(z) b_\tau^\top(z)). \tag{38}$$

Based on the definition of the moment and localizing matrices, the following theorem, which is the basis for the Lasserre's hierarchy [5], can be stated.

Theorem 4 [42, Sec. 4.1.5] *If $m = \{m_\alpha\}_{\alpha \in \mathcal{A}_{2\tau}^{n_z}}$ is a sequence of moments generated by a probability measure $\Pr_z(z)$ supported by $\boldsymbol{Z}$, then*

$$M_\tau(m) \succeq 0, \;\; m_{0\cdots 0} = 1, \;\; M_{\tau - \left\lceil \frac{deg(q_j)}{2} \right\rceil}(q_j m) \succeq 0, j = 1, \dots n_q, \tag{39}$$

for any integer $\tau \geq \tilde{\tau}$, with $\tilde{\tau}$ defined in (34). ■

Proof irst, observe that if $m = \{m_\alpha\}_{\alpha \in \mathcal{A}_{2\tau}^{n_z}}$ is a sequence of moments generated by a probability measure $\Pr_z$ supported by $\boldsymbol{Z}$, then:

$$m_\alpha = \int z^\alpha dP_z(z), \quad m_{0\cdots 0} = \int_{\boldsymbol{Z}} dP_z(z) = 1.$$

Based on the definition of the moment matrix $M_\tau(m)$ (see (37)), for any real vector $\boldsymbol{g}$ of proper dimension, we have

$$\begin{aligned} \boldsymbol{g}^\top M_\tau(m) \boldsymbol{g} &= \int \boldsymbol{g}^\top b_\tau(z) b_\tau^\top(z) \boldsymbol{g} dP_z(z) \\ &= \int g^2(z) dP_z(z) \geq 0, \end{aligned} \tag{40}$$

where $g(z)$ is a generic polynomial in $\mathbb{R}_\tau$, whose vector of coefficients in the canonical basis $b_\tau(z)$ is $\boldsymbol{g}$. Since condition (40) holds for any vector $\boldsymbol{g}$, $M_\tau(m) \succeq 0$.

For any $j = 1, \ldots, n_q$, let us now take another real-valued vector $\boldsymbol{g}$ of proper dimension, and consider the term

$$\boldsymbol{g}^\top M_{\tau - \lceil \frac{deg(q_j)}{2} \rceil}(q_j m)\boldsymbol{g}. \tag{41}$$

Based on the definition of the localizing matrix $M_{\tau - \lceil \frac{deg(q_j)}{2} \rceil}(q_j m)$ (see eq. (38)), the term (41) becomes:

$$\boldsymbol{g}^\top M_{\tau - \lceil \frac{deg(q_j)}{2} \rceil}(q_j m)\boldsymbol{g} = \int q_j(z)\boldsymbol{g}^\top b_\tau(z) b_\tau^\top(z)\boldsymbol{g} dP_z(z)$$
$$= \int q_j(z) g^2(z) dP_z(z) = \int_{\boldsymbol{Z}} q_j(z) g^2(z) dP_z(z) \geq 0, \tag{42}$$

where the above inequality holds since, by definition of the set $\boldsymbol{Z}$ (eq. (32)), $q_j(z) \geq 0$ for any $z \in \boldsymbol{Z}$. Thus, $M_{\tau - \lceil \frac{deg(q_j)}{2} \rceil}(q_j m) \succeq 0$.

Based on Theorem 4, for any integer $\tau \geq \tilde{\tau}$, instead of requiring the conditions in (36), one may require the weaker conditions in (39). This leads to an upper bound $\overline{p}^\tau$ of $\bar{p}$, which can be computed by solving the (convex) SDP problem:

$$\overline{p}^\tau = \sup_{m = \{m_\alpha\}_{\alpha \in \mathcal{A}_{2\tau}^{n_z}}} \sum_{\alpha \in \mathcal{A}_{2\tau}^{n_z}} \boldsymbol{h}_\alpha m_\alpha \tag{43a}$$

$$s.t.$$

$$\sum_{\alpha \in \mathcal{A}_{2\tau}^{n_z}} \tilde{\boldsymbol{f}}_{i,\alpha} m_\alpha = \mu_i, \quad i = 2, \ldots, n_f, \tag{43b}$$

$$m_{0\cdots 0} = 1, \quad M_\tau(m) \succeq 0, \tag{43c}$$

$$M_{\tau - \lceil \frac{deg(q_j)}{2} \rceil}(q_j m) \succeq 0, \quad j = 1, \ldots n_q. \tag{43d}$$

Example 1.9 *In the explanatory example considered so far,*

$$h(z) = x_1^2 + x_2^2 = z_3^2 + z_4^2 = z^{0020} + z^{0002},$$
$$\tilde{f}_2(z) = \rho = z_1 = z^{1000}.$$

Thus, for a relaxation order $\tau = 2$, the SDP problem (43) is given by:

$$\overline{p}^\tau = \sup_{m = \{m_\alpha\}_{\alpha \in \mathcal{A}_{2\tau}^4}} m_{0020} + m_{0002}$$
$$s.t.\ m_{0000} = 1, \quad m_{1000} = 0.5$$
$$M_1(m) \succeq 0, \quad M_0(q_j m) \succeq 0, \quad j = 1, \ldots 7, \tag{44}$$

with

$$M_1(m) = \begin{bmatrix} m_{0000} & m_{1000} & m_{0100} & m_{0010} & m_{0001} \\ m_{1000} & m_{2000} & m_{1100} & m_{1010} & m_{1001} \\ m_{0100} & m_{1100} & m_{0200} & m_{0110} & m_{0101} \\ m_{0010} & m_{1010} & m_{0110} & m_{0020} & m_{0011} \\ m_{0001} & m_{1001} & m_{0101} & m_{0011} & m_{0002} \end{bmatrix},$$

$$
\begin{aligned}
&M_0(q_1 m) = m_{1000}, \quad M_0(q_2 m)=1-m_{1000}, \quad M_0(q_3 m)=m_{0100},\\
&M_0(q_4 m) = m_{1010} - m_{0010}, \quad M_0(q_5 m) = -m_{1010} + m_{0010},\\
&M_0(q_6 m) = m_{0001}, \quad M_0(q_7 m) = -m_{0001},\\
&M_0(q_8 m) = 1 - m_{0020} - m_{0002}.
\end{aligned}
$$

■

By construction, the moment and the localizing matrices are such that:

$$
\begin{aligned}
M_{\tau+1}(m) \succeq 0 &\Rightarrow M_\tau(m) \succeq 0,\\
M_{\tau+1-\left\lceil \frac{deg(q_j)}{2} \right\rceil}(q_j m) \succeq 0 &\Rightarrow M_{\tau-\left\lceil \frac{deg(q_j)}{2} \right\rceil}(q_j m) \succeq 0.
\end{aligned}
$$

This implies:

$$
\bar{p}^\tau \geq \bar{p}^{\tau+1} \geq \bar{p}, \tag{45}
$$

which means that, as the relaxation order τ increases, the SDP relaxation (43) becomes tighter. Furthermore, under mild restrictive assumptions on the description of the set $\boldsymbol{Z}$, the solution of the SDP relaxed problem (43) converges to the global optimum $\bar{p}$ of the original optimization problem (30), i.e.,

$$
\lim_{\tau \to \infty} \bar{p}^\tau = \bar{p}. \tag{46}
$$

The proof of the converge property in (46) is reported in the appendix, along with the needed assumptions.

Remark 1 *The number N_τ of the optimization variables $m = \{m_\alpha\}_{\alpha \in \mathcal{A}_{2\tau}^{n_z}}$ of problem* (43) *is given by the binomial expression:*

$$
N_\tau = \left(\begin{array}{c} n_z + 2\tau \\ 2\tau \end{array} \right) = O\left(n_z^{2\tau}\right),
$$

and thus, for fixed relaxation order τ, N_τ grows polynomially with the size of the vector z.

Property 1 *Since the relaxed SDP problem* (43) *provides an upper bound of $\bar{p}$ (i.e., $\bar{p}^\tau \geq \bar{p}$), sufficient conditions on the $\mathcal{D}$-stability of $A(\rho)$ can be derived from $\bar{p}^\tau$. Specifically:*

- *if the only information on the uncertain parameter vector ρ is the support Δ of its probability measures (i.e., $\rho \in \Delta$), then, from Theorem 2 and Corollary 1, $\bar{p}$ can be either* 0 *($A(\rho)$ is robustly $\mathcal{D}$-stable) or* 1 *($A(\rho)$ is not robustly $\mathcal{D}$-stable). Thus, if $\bar{p}^\tau < 1$, we can claim that $\bar{p} = 0$ and thus $A(\rho)$ is guaranteed to be robustly $\mathcal{D}$-stable against the uncertainty set Δ. On the other hand, if $\bar{p}^\tau \geq 1$, no conclusions can be drawn, in principle, on the robust $\mathcal{D}$-stability of $A(\rho)$.*

- *if the information on the moments of ρ are given, then $\bar{p}$ represents the probability of the matrix $A(\rho)$ to have at least an eigenvalue in $\mathcal{D}^c$. Thus, since $\bar{p}^\tau \geq \bar{p}$, we can claim that $A(\rho)$ is not $\mathcal{D}$-stable with probability less than or equal to $\bar{p}^\tau$. Equivalently, $A(\rho)$ is $\mathcal{D}$-stable with probability at least $1 - \bar{p}^\tau$.* ■

Example 1.10 *Let us go back to the explanatory example. For a relaxation order* $\tau = 2$*, the solution of the SDP problem* (44) *is* $\overline{p}^{\tau} = 0.5$*. Thus, we can claim that* $A(\rho)$ *is not* $\mathcal{D}$*-stable with probability at most* $= 0.5$*. Note that the obtained solution* $\overline{p}^{\tau} = 0.5$ *is tight (i.e.,* $\overline{p}^{\tau} = \overline{p}$*). In fact, we have already seen in Example 1.7 that, for a probability measure* $\Pr_{\rho} = 0.5\delta_{(0)} + 0.5\delta_{(1)}$*, the matrix* $A(\rho)$ *has an eigenvalue equal to* 0 *with probability* 0.5. ■

6 Applications and examples

In this section, we show the application of the proposed approach through three numerical examples. The problem of robust Hurwitz stability analysis of uncertain matrices is addressed in the first example, and a comparison with the polynomial optimization based approaches proposed in [40, 39] is also provided. Robust and probabilistic analysis of the properties of dynamical models with parametric uncertainty is discussed in the second and in the third examples. Specifically, in the second example, taken from [25], sufficient conditions for nonexistence of bifurcations in uncertain nonlinear continuous-time dynamical systems are derived. Both the deterministic and the probabilistic scenario are considered. The other example is focused on the analysis of robust stability and performance verification of LTI systems with parametric uncertainty. The robust and probabilistic formulations are combined to verify robust stability of the system and to compute the minimum probability to meet the performance specifications.

All computations are carried out on an i7 2.40-GHz Intel core processor with 3 GB of RAM running MATLAB R2014b. The YALMIP Matlab interface [43] is used to construct the relaxed SDP problems (43), which are solved through the general purpose SDP solver SeDuMi [44].

6.1 Hurwitz stability and polynomial abscissa

The aim of this example is to highlight the advantages of our approach w.r.t. the polynomial optimization based methods presented in [40, 39]. Since the method in [40] is focused on the approximation of the abscissa of an uncertain polynomial (i.e., maximum real part of the roots of a univariate polynomial), a robust Hurwitz stability analysis problem is discussed.

Let us consider the uncertain matrix

$$A(\rho) = \begin{bmatrix} -2.4 - \rho_1^2 & 6 - \rho_1^2 \\ 1 - 2\rho_1^2 & -2.9 - 2\rho_1 \end{bmatrix}, \tag{47}$$

with $\rho_1 \in \Delta = [-0.1 \;\; 3.4]$, whose characteristic polynomial is given by:

$$\begin{aligned} P(s, \rho_1) = & s^2 + (5.3 + 2\rho_1 + \rho_1^2)s \\ & +0.96 + 4.8\rho_1 + 15.9\rho_1^2 + 2\rho_1^3 - 2\rho_1^4. \end{aligned} \tag{48}$$

<u>*Polynomial abscissa approximation [40]*</u>

The main idea in [40] is to find a fixed-degree polynomial $\overline{P}_d(\rho_1)$ approximating, from above, the abscissa $a(\rho_1)$ of the polynomial $P(s, \rho_1)$. Specifically, among all the polynomials $\overline{P}_d(\rho_1)$ of given degree d such that

$$\overline{P}_d(\rho_1) \geq a(\rho_1) \;\; \forall \rho_1 \in \Delta,$$

the one minimizing the integral

$$\int_{\rho_1 \in \Delta} \overline{P}_d(\rho_1) d\rho_1 \tag{49}$$

is sought. SDP relaxations based on *sum-of-squares* are then used to find the upper approximating polynomial $\overline{P}_d(\rho_1)$. Note that, if $\max_{\rho_1 \in \Delta} \overline{P}_d(\rho_1) < 0$, then all the roots of the characteristic polynomial $P(s, \rho_1)$ have negative real part, thus the matrix $A(\rho)$ in (47) is guaranteed to be robust Hurwitz stable. Fig. 2 shows the abscissa $a(\rho_1)$ of the polynomial $P(s, \rho_1)$, along with computed upper approximating polynomial $\overline{P}_d(\rho_1)$ of degree $d = 8$ in the interval $\Delta = [-0.1 \;\; 3.4]$. As $\overline{P}_d(\rho_1) \geq 0$ for some values of $\rho_1 \in \Delta$, no conclusions can be drawn from $\overline{P}_d(\rho_1)$ on robust Hurwitz stability of the matrix $A(\rho)$. This conservativeness is due to the fact that the computed polynomial $\overline{P}_d(\rho_1)$ is the "best" (w.r.t. the integral (49)) upper approximation of the abscissa $a(\rho_1)$ over the whole uncertainty set Δ. On the other hand, in assessing robustly Hurwitz stability of $P(s, \rho_1)$, we are only interested in approximating the maximum of the abscissa over $\rho_1 \in \Delta$. The CPU time required to verify Hurwitz stability of $A(\rho)$ is 2.5 s. This includes the time required to compute the upper approximating polynomial $\overline{P}_d(\rho_1)$ as well as the time required to compute its maximum over ρ_1 through Lasserre's relaxation.

Note that, in the general case of multidimensional uncertainty ρ, another source of conservativeness may also come from the fact that the maximum of the polynomial $\overline{P}_d(\rho)$ over Δ cannot be computed with a simple plot, but it should be computed through the Lasserre's SDP relaxation [5], which only provides an upper bound of the maximum of $\overline{P}_d(\rho)$. Finally, in case the polynomial $\overline{P}_d(\rho)$ is of large degree (say, $d > 10$), a large Lasserre's relaxation order may be needed to achieve a tight approximation of the maximum of $\overline{P}_d(\rho)$, thus leading to Lasserre's relaxations which might be computationally intractable.

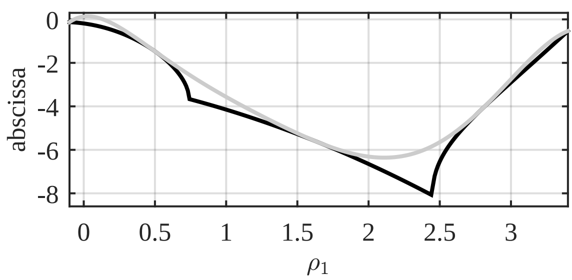

Figure 2: Abscissa of the polynomial $P(s, \rho_1)$ in (48) (black line) and computed polynomial approximation (gray line).

Hermite stability criterion [39]
The problem of robust $\mathcal{D}$-stability of a polynomial is tackled in [39] approximating the minimum eigenvalue of the associated Hermite matrix. In order to check Hurwitz stability of the polynomial $P(s, \rho_1)$ in (48), the associated

2×2 symmetric Hermite matrix $H(\rho_1)$ is constructed. Since the coefficients of $P(s, \rho_1)$ are polynomials in ρ_1 of maximum degree 4, the entries of the matrix $H(\rho_1)$ are polynomials in ρ_1 of maximum degree 8. According to the Hermite stability criterion (see [39]), $P(s, \rho_1)$ is robustly Hurwitz stable if and only if

$$H(\rho_1) \succ 0, \quad \forall \rho_1 \in \Delta. \tag{50}$$

The robust minimum eigenvalue of $H(\rho_1)$ is given by

$$\lambda_{\min} = \min_{\rho \in \Delta} \min_{x \in \mathbb{R}^2 : x^\top x = 1} x^\top H(\rho_1) x. \tag{51}$$

As well known, (50) holds, or equivalently $P(s, \rho_1)$ is robustly Hurwitz stable, if and only if $\lambda_{\min} > 0$. Then, a lower bound $\underline{\lambda}_{\min}$ of $\lambda_{\min}$ is computed solving the polynomial optimization problem (51) through the Lasserre's hierarchy, for a relaxation order $\tau = 5$, which is the minimum allowed value for τ, as the objective function in (51) is a 10-degree polynomial in the augmented variable $[x \ \rho_1]$. We obtain a lower bound $\underline{\lambda}_{\min} = 6.5$, in a CPU time of 2.7 s. The obtained results allow us to claim that $H(\rho_1)$ is robustly positive definite, thus $P(s, \rho_1)$ is robustly Hurwitz stable, and and no conservativeness is introduced in relaxing problem (51) through the Lasserre's hierarchy. However, the example shows that even if the entries of the matrix $A(\rho)$ are polynomial functions of ρ_1 of degree at most 2, the objective function minimized in (51) is a polynomial of degree 10, which required to use a Lasserre's relaxation order at least equal to $\tau = 5$. As already discussed, the Lasserre's hierarchy may become computationally intractable in the more general case of multidimensional uncertain parameter ρ and large relaxation orders.

Robust $\mathcal{D}$-stability analysis

The approach proposed in this paper is now used to assess robust Hurwitz stability of the matrix $A(\rho)$. The polynomial optimization problem (16) is formulated, and solved through the Lasserre's hierarchy for a relaxation order $\tau = 3$ (namely, the SDP problem (43) is solved without using any information on the moments of ρ_1). The obtained solution of the SDP relaxed problem (43) is 10^{-9}. Thus, according to Property 1, $A(\rho)$ is robustly Hurwitz stable. The CPU time required to assess robust Hurwitz stability of $A(\rho)$ is 1.5 s. Thus, in this simple example, the proposed approach is about 1.6x faster than the methods [39] and [40]. This is due to the fact that, in the presented approach, the Lasserre's relaxation order τ can be kept "small", as the maximum degree of the polynomial constraints in (16) is 3 because of the product $A(\rho)x$.

6.2 Bifurcation analysis

The example discussed in this section has been recently studied in [25], where the analysis of the location of the eigenvalues of an uncertain matrix is applied to derive sufficient conditions for nonexistence of bifurcations in nonlinear continuous-time dynamical systems with parametric uncertainty.

As an example, [25] considers a continuous-time predator-prey model, described by the differential equations

$$\dot{r}_1 = \gamma r_1(1-r_1) - \frac{\rho_1 r_1 r_2}{\rho_2 + r_1}, \tag{52a}$$

$$\dot{r}_2 = -\rho_3 r_2 + \frac{\rho_1 r_1 r_2}{\rho_2 + r_1}, \tag{52b}$$

where r_1 and r_2 are scaled population numbers, $\gamma = 0.1$ is the prey growth rate, ρ_1, ρ_2 and ρ_3 are real uncertain parameters.

A non-trivial equilibrium point for the model (52) is:

$$r_{1,\mathrm{eq}} = \frac{\rho_2\rho_3}{\rho_1-\rho_3}, \quad r_{2,\mathrm{eq}} = \frac{\gamma\rho_2}{\rho_1-\rho_3}\left(1 - \frac{\rho_2\rho_3}{\rho_1-\rho_3}\right). \tag{53}$$

The Jacobian J of the system at the equilibrium point $(r_{1,\mathrm{eq}}, r_{2,\mathrm{eq}})$ in (53) is

$$J(r_{1,\mathrm{eq}}, r_{2,\mathrm{eq}}) = \begin{bmatrix} \gamma\frac{\rho_3}{\rho_1}\left(1-\rho_2\frac{\rho_1+\rho_3}{\rho_1-\rho_3}\right) & -\rho_3 \\ \gamma\frac{1}{\rho_1}\left(\rho_1-\rho_3-\rho_2\rho_3\right) & 0 \end{bmatrix}. \tag{54}$$

Well known results from the bifurcation theory [45] state that a sufficient condition to guarantee the existence of no local bifurcations at the equilibrium point $(r_{1,\mathrm{eq}}, r_{2,\mathrm{eq}})$ is that $J(r_{1,\mathrm{eq}}, r_{2,\mathrm{eq}})$ has no eigenvalues with zero real part.

Let us consider uncertain parameters ρ_1, ρ_2 and ρ_3 which take values in the intervals

$$\rho_i \in \left[\rho_i^{\mathrm{o}} - k\Delta\rho_i \quad \rho_i^{\mathrm{o}} + k\Delta\rho_i\right], \quad i = 1,2,3, \tag{55}$$

where ρ_i^{o} denotes the nominal value of the parameter ρ_i, $k \in \mathbb{R}$ is a scaling factor, and $\Delta\rho_i$ characterizes the width of the uncertainty interval where ρ_i belongs to. Like in [25], we assume that the uncertainty intervals in (55) share the same width, i.e., $\Delta\rho_i = 1$ for all $i = 1,2,3$, and they are centered at the nominal values $\rho_1^{\mathrm{o}} = 9$, $\rho_2^{\mathrm{o}} = 2$ and $\rho_3^{\mathrm{o}} = 2$.

Note that the entries of the Jacobian $J(r_{1,\mathrm{eq}}, r_{2,\mathrm{eq}})$ are not polynomial functions in the uncertain parameters ρ_1, ρ_2 and ρ_3. However, by introducing the slack variables:

$$t_1 = \frac{\rho_3}{\rho_1}, \quad t_2 = \frac{1}{\rho_1-\rho_3},$$

the entries of the matrix $J(r_{1,\mathrm{eq}}, r_{2,\mathrm{eq}})$ can be rewritten as polynomial functions in ρ_1, ρ_2, ρ_3, t_1, t_2, i.e.,

$$J(r_{1,\mathrm{eq}}, r_{2,\mathrm{eq}}) = \begin{bmatrix} \gamma t_1\left(1-\rho_2\left(\rho_1+\rho_3\right)t_2\right) & -\rho_3 \\ \gamma\left(1-t_1-\rho_2 t_1\right) & 0 \end{bmatrix},$$

where the additional polynomial constraints:

$$\rho_1 t_1 = \rho_3, \quad (\rho_1-\rho_3)t_2 = 1, \tag{56}$$

have to be considered along with the interval constraints (55) on ρ_i to maintain the relationship among the entries of the matrix $J(r_{1,\mathrm{eq}}, r_{2,\mathrm{eq}})$. This leads to an augmented set of uncertain variables (namely, $\rho_1, \rho_2, \rho_3, t_1, t_2$), which are constrained to belong to the nonconvex uncertainty set described by the constraints

(55) and (56).

Deterministic bifurcation analysis
Let $\mathcal{D}^c$ be the imaginary axis of the complex place. i.e.,

$$\mathcal{D}^c = \{\lambda \in \mathbb{C} : \lambda = \lambda_{\text{re}} + j\lambda_{\text{im}}, \quad \lambda_{\text{re}}, \lambda_{\text{im}} \in \mathbb{R}, \; \lambda_{\text{re}} = 0\},$$

For fixed width k of the uncertainty intervals $[\rho_i^{\text{o}} - k\Delta\rho_i \;\; \rho_i^{\text{o}} + k\Delta\rho_i]$, the deterministic bifurcation analysis problem can be formulated as a $\mathcal{D}$-stability analysis problem, or equivalently, in terms of problem (18), by assuming to know only the support of the uncertain parameters ρ_1, ρ_2, ρ_3. An upper bound $\overline{p}^\tau$ of $\overline{p}$ (i.e., solution of (18)) is computed by solving the relaxed SDP problem (43) for a relaxation order $\tau = 3$.

Based on considerations given in Property 1, if $\overline{p}^\tau < 1$, then $J(r_{1,\text{eq}}, r_{2,\text{eq}})$ is guaranteed to have no eigenvalues on the imaginary axis for any $\rho_i \in [\rho_i^{\text{o}} - k\Delta\rho_i \;\; \rho_i^{\text{o}} + k\Delta\rho_i]$. A bisection on the width k of the uncertainty intervals $[\rho_i^{\text{o}} - k\Delta\rho_i \;\; \rho_i^{\text{o}} + k\Delta\rho_i]$ is then carried out to compute (a lower bound of) the maximum value of k such that $J(r_{1,\text{eq}}, r_{2,\text{eq}})$ is guaranteed not to have any eigenvalues on the imaginary axis for any $\rho_i \in [\rho_i^{\text{o}} - k\Delta\rho_i \;\; \rho_i^{\text{o}} + k\Delta\rho_i]$. The obtained value of k is $k = 0.4620$ (similar to the result obtained in [25]) and the CPU time required to solve problem (43) for fixed k is, in average, 536 seconds. Since sufficient conditions on robust $\mathcal{D}$-stability are derived from $\overline{p}^\tau$, we can claim that the system is guaranteed to have no local bifurcation at the equilibrium point $(r_{1,\text{eq}}, r_{2,\text{eq}})$ for any ρ_i in the interval $[\rho_i^{\text{o}} - k\Delta\rho_i \;\; \rho_i^{\text{o}} + k\Delta\rho_i]$, with $i = 1, 2, 3$ and $k = 0.4620$.

In this example, tightness of the computed solution can be verified analytically. In fact, the determinant of $J(r_{1,\text{eq}}, r_{2,\text{eq}})$ is:

$$det(J(r_{1,\text{eq}}, r_{2,\text{eq}})) = \alpha \frac{\rho_3}{\rho_1} (\rho_1 - \rho_3 - \rho_2\rho_3), \tag{57}$$

which is equal to zero for $\rho_1 = 8.5412$, $\rho_2 = \rho_3 = 2.4650$. This values of ρ_1, ρ_2 and ρ_3 lie in the intervals $[\rho_i^{\text{o}} - k\Delta\rho_i \;\; \rho_i^{\text{o}} + k\Delta\rho_i]$ for $k = 0.4650$.

Probabilistic bifurcation analysis
Let us now consider the case where the uncertain parameters ρ_i belong to the intervals

$$\rho_i \in [\rho_i^{\text{o}} - k\Delta\rho_i \;\; \rho_i^{\text{o}} + k\Delta\rho_i], \quad i = 1, 2, 3,$$

with $k = 1$. The expected values of all the three uncertain parameters are known and equal to their nominal values, i.e.,

$$\mathbb{E}[\rho_i] = \rho_i^{\text{o}}, \quad i = 1, 2, 3.$$

Furthermore, we assume that an upper bound $\overline{\sigma}^2$ on the variance of the probability measure Pr_ρ describing the parameters ρ_i is known, i.e.,

$$\int_\Delta (\rho_i - \rho_i^{\text{o}})^2 \, dP_\rho(\rho) \le \overline{\sigma}^2, \quad i = 1, 2, 3.$$

The solution $\overline{p}^\tau$ of the corresponding SDP problem (43) is computed for a relaxation order $\tau = 3$ and for different values of the (upper bound on the) variance $\overline{\sigma}^2$. Fig. 3 shows the computed upper probability $\overline{p}^\tau$ of the system to

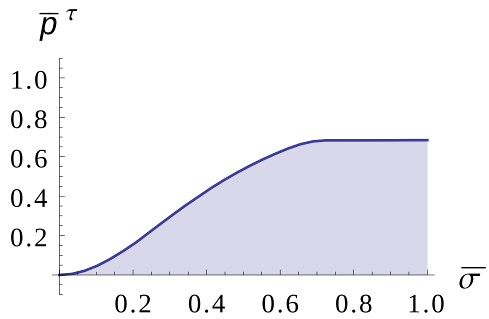

Figure 3: Bifurcation analysis: maximum standard deviation $\overline{\sigma}$ of the probability measures Pr_{ρ} vs. upper probability $\overline{p}^{\tau}$ of having a local bifurcation.

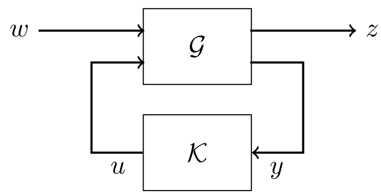

Figure 4: Feedback control system. $\mathcal{G}$: plant; $\mathcal{K}$: controller; w: generalized disturbance; u: control input; z: controlled output; y: measured output.

have a local bifurcation at the equilibrium point $(r_{1,\mathrm{eq}}, r_{2,\mathrm{eq}})$ for different values of the variance $\overline{\sigma}^2$. It can be observed that, although for a width $k = 1$ of the uncertainty intervals the system is not guaranteed to have no local bifurcation, under the considered assumptions on the mean and the maximum variance, the probability that the system has a local bifurcation at the equilibrium point $(r_{1,\mathrm{eq}}, r_{2,\mathrm{eq}})$ is, in the worst-case scenario, smaller than 0.1 for $\overline{\sigma}^2$ smaller than 0.15^2. *In other words the system has not local bifurcation with probability at least* 0.9. Therefore with such an information on the moments we can guarantee that the system has not local bifurcations with probability at least 0.9, considering an interval width $k = 1$ (that is more than two times the one considered in the deterministic case ($k = 0.462$)). *We can thus be much less conservative and at the same guaranteeing no bifurcation with "high probability".* Note also that, for values of $\overline{\sigma}^2$ larger than 0.67^2, the (upper) probability $\overline{p}^{\tau}$ of having a local bifurcation saturates to 0.68. This seems to indicate that, above a threshold $\overline{\sigma}^2 = 0.67^2$, the probability of having a local bifurcation does not increase as the set of feasible probability measures Pr_{ρ} enlarges.

6.3 Robust stability and performance analysis of LTI systems

In this example, we show how the proposed approach can be used to check robust stability and (probabilistic) satisfaction of performance requirements in uncertain LTI systems.

Consider the closed-loop system depicted in Fig. 4. The state-space representation of the plant $\mathcal{G}$ is given by:

$$\dot{x} = Ax + B_u u + B_w w, \tag{58a}$$

$$\begin{bmatrix} z \\ y \end{bmatrix} = \begin{bmatrix} C_z \\ C_y \end{bmatrix} x, \tag{58b}$$

where $x = [x_1\ x_2\ x_3\ x_4]^\top$ denotes the state of the system, $y = [y_1\ y_2\ y_3\ y_4]^\top$ is the measured output that enters the controller $\mathcal{K}$, u, w and z are the control input, generalized disturbance and the controlled output, respectively. The values of the matrices in (58) are:

$$A = \begin{bmatrix} 0 & 1+0.2\rho_1-0.1\rho_2 & -0.5 & 3\rho_3 \\ \rho_2 & -0.2+0.1\rho_3-0.3\rho_1 & -0.4 & -10 \\ -4 & -0.1+\rho_4-0.5\rho_2 & -0.5 & 1.5 \\ 0.4+0.2\rho_2\rho_3 & 3 & 4+0.5\rho_1 & 1+\rho_4^2 \end{bmatrix}$$

$$B_u = \begin{bmatrix} 1\,1\,0\,1 \end{bmatrix}^T,\ B_w = \begin{bmatrix} 1.25\ 1.25\ 1.25\ 1.25 \end{bmatrix}^T,$$
$$C_z = \begin{bmatrix} 1.25 & 0 & 0 & 0 \end{bmatrix},\ C_y = \mathrm{diag}(\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}).$$

The parameters ρ_1, ρ_2, ρ_3 and ρ_4 defining the dynamic matrix A are not known exactly and they belong to the uncertainty intervals

$$\rho_1 \in [\rho_1^o - 0.15\ \ \rho_1^o + 0.15],\ \rho_2 \in [\rho_2^o - 0.05\ \ \rho_2^o + 0.05], \tag{59a}$$

$$\rho_3 \in [\rho_3^o - 0.25\ \ \rho_3^o + 0.25],\ \rho_4 \in [\rho_4^o - 0.05\ \ \rho_4^o + 0.05], \tag{59b}$$

where $\rho_1^o = 1$, $\rho_2^o = 0$, $\rho_3^o = 0$ and $\rho_4^o = 0$ are the nominal values of the parameters.

The controller $\mathcal{K}$ is a static output-feedback controller (i.e., $u = -Ky = -Kx$) designed to place the poles of the nominal closed-loop system at $-0.5 \pm j$, -5 and -5. This is achieved for a matrix gain $K = [36.45\ \ -5.33\ \ -30.67\ \ -11.12]$.

In order to verify the robust stability of the closed-loop system, we check if the (uncertain) closed-loop dynamic matrix

$$A_{\mathrm{cl}} = A - B_u K$$

has no eigenvalues with positive or null real part. This equivalent to verify that the solution $\bar{p}$ of the optimization problem (18) is 0, where the only information used in (18) is the uncertainty intervals where the parameters ρ_1, ρ_2, ρ_3 and ρ_4 are supposed to belong to, and $\mathcal{D}^c$ is the closed right-half plane of the complex plane. Thus, based on the considerations in Property 1, a sufficient condition to guarantee that $\bar{p} = 0$ (or equivalently, the system is robustly stable) is $\overline{p}^\tau < 1$. By solving problem (43) for a relaxation order $\tau = 2$, we obtain $\overline{p}^\tau = 0.05$ (CPU time=44.98 seconds), thus proving robust stability of the closed-loop system.

Robust and probabilistic performance analysis

Like in H_∞-control design, the performance of the closed-loop system are specified in terms of the H_∞-norm of the closed-loop system $\mathcal{G}_{\mathrm{cl}}$ relating the generalized disturbance w and the controlled output z, whose state-space representation is given by:

$$\dot{x} = (\underbrace{A - B_u K}_{A_{\rm cl}})x + B_w w,$$
$$z = C_z \; x.$$

For a given $\eta > 1$, we claim that robust performance is achieved if

$$\|\mathcal{G}_{\rm cl}\|_\infty < \eta,$$

for all values taken by the parameters $\rho_1, \rho_2, \rho_3, \rho_4$ in the uncertainty intervals in (59).

As well known in the H$_\infty$-control theory, the condition $\|\mathcal{G}_{\rm cl}\|_\infty < \eta$ holds if and only if the *Hamiltonian* matrix

$$H = \left[\begin{array}{cc} A_{\rm cl} & \frac{1}{\eta^2} B_w B_w^\top \\ -C_z C_z^\top & -A_{\rm cl}^\top \end{array} \right]$$

has no eigenvalues on the imaginary axis. Let us set $\eta = 1$. By solving the corresponding SDP relaxed problem (43) for a relaxation order $\tau = 2$, we obtain $\overline{p}^\tau = 1$. Thus, in principle, we cannot draw any conclusions on the robust performance of the system.

Nevertheless, some heuristics can be used to verify, from the solution of problem (43), if the Hamiltonian H has some eigenvalues on the imaginary axis. In fact, when Lasserre's hierarchy is used to relax (deterministic) polynomial optimization problems (like (16)), the first order moments m_α of the SDP relaxed problem (43) provides, in practice, a good approximation $(\hat{\rho}, \hat{x}, \hat{\lambda})$ of the global minimizer (ρ^*, x^*, λ^*) of the original optimization problem (16). By looking at the first order moments m_α associated to the uncertain parameters ρ, we obtain

$$\hat{\rho} = [\hat{\rho}_1 \;\; \hat{\rho}_2 \;\; \hat{\rho}_3 \;\; \hat{\rho}_4]^T = [1.101 \;\; 0.047 \;\; -0.222 \;\; -0.005]^T .$$

For this values of the uncertainty ρ, we obtain $\|\mathcal{G}_{\rm cl}\|_\infty = 1.013$. Thus, we can claim that robust performance requirements are not achieved.

Finally, the probabilistic framework is considered. Probabilistic conditions on the performance of the system are derived under the assumption that the expected value of the uncertain parameters is given by their nominal parameters $\rho_1^{\rm o} = 1$, $\rho_2^{\rm o} = \rho_3^{\rm o} = \rho_4^{\rm o} = 0$, and the maximum variance $\overline{\sigma}_i^2$ of the probability measures ${\rm Pr}_{\rho_i}$ describing the uncertain parameters ρ_1, ρ_2, ρ_3 and ρ_4 is available. Specifically,

$$\int_\Delta (\rho_i - \rho_i^{\rm o})^2 \, dP_{\rho_i}(\rho_i) \leq \overline{\sigma}_i^2, \tag{60}$$

with $\overline{\sigma}_1 = 0.024, \overline{\sigma}_2 =, 0.008, \overline{\sigma}_3 = 0.040, \overline{\sigma}_4 = 0.008$.

Solving the corresponding SDP problem (43) for a relaxation order $\tau = 2$, we obtain $\overline{p}^\tau = 0.082$ (CPU time=4916 seconds). Based on the obtained results, *we can claim that the closed-loop system is guaranteed to be robustly stable and the performance requirements are fulfilled with probability at least* 0.918.

For a more exhaustive analysis on the performance of the system, we also compute the (minimum) probability $\underline{p}^\tau = 1 - \overline{p}^\tau$ to satisfy the condition $\|\mathcal{G}_{\rm cl}\|_\infty <$

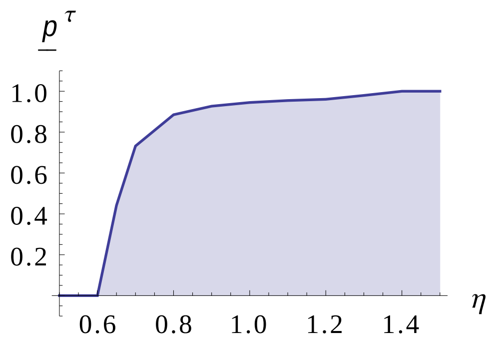

Figure 5: Probabilistic performance analysis: norm bound η vs lower bound $\underline{p}^\tau$ on $\Pr_\rho(\|G_{\rm cl}\|_\infty < \eta)$.

η for different values of η. The obtained results are reported in Fig. 5, which shows the computed $\underline{p}^\tau$ (representing a lower bound on the probability $\Pr_\rho(\|G_{\rm cl}\|_\infty < \eta)$) w.r.t. different values of the norm bound η. Note that, for $\eta \geq 1.4$, the constraint $\|\mathcal{G}_{\rm cl}\|_\infty < \eta$ is guaranteed to be satisfied with probability 1, which also means (based on Theorem 1) that $\|\mathcal{G}_{\rm cl}\|_\infty < \eta$ for all uncertain parameters $\rho_1, \rho_2, \rho_3, \rho_4$ in the considered uncertainty intervals.

7 Conclusions

In this paper, we have presented a unified framework for deterministic and probabilistic analysis of $\mathcal{D}$-stability of uncertain matrices. A family of matrices whose members have entries which vary in an uncertainty set described by polynomial constraints is considered, and stability regions $\mathcal{D}$ whose complement is described by polynomial constraints can be handled. This class of stability sets is quite vast and includes, among others:

- the open left half plane and the unit circle of the complex plane, which allows us to verify stability of continuous- and discrete-time LTI systems with parametric uncertainty;
- the imaginary axis, which allows us to compute an upper bound on the H_∞-norm of uncertain LTI systems;
- the semi-axis of positive real numbers, which allows us to verify robust and probabilistic positive definiteness of a family of real symmetric matrices;
- the origin of the complex plane, which allows us to verify robust and probabilistic nonsingularity of uncertain matrices.

Actually, the approach described in the paper is widely applicable and it can undoubtedly be used to tackle many problems in systems and control theory.

The rationale behind the method is to formulate a generalized moment optimization problem which is relaxed through the Lasserre's hierarchy into a sequence of *semidefinite programming* (SDP) problems of finite size. The relaxed problems provide lower bounds on the minimum probability of a family of matrices to be $\mathcal{D}$-stable. This is equivalent, in the deterministic realm, to derive sufficient conditions for robust $\mathcal{D}$-stability. It has been observed that, in practice, the level of conservativeness due to the Lasserre's relaxation is relatively "small", and tight solutions are obtained in many cases.

The deterministic and the probabilistic analysis can also be easily combined to handle scenarios where some parameters are only known to vary within given uncertainty regions, and other parameters are also characterized by probabilistic information (like mean or variance).

Future activities will be devoted to extending the ideas underlying the developed method to both robust and probabilistic control synthesis problems. Furthermore, in order to reduced the computational time required in solving the relaxed SDP problems, dedicated numerical algorithms will be developed, thus avoiding the use of general purpose SDP solvers.

Convergence of the Lasserre's hierarchy

In this section, we discuss convergence of the solution $\overline{p}^{\tau}$ of the SDP relaxed problem (43) to the global optimum $\overline{p}$ of problem (30). First, some useful lemmas and results are given.

Lemma 1 [**Putinar's representation of positive polynomials over semi-algebraic sets [46]**]

Suppose that the set $\boldsymbol{Z}$ in (32) is compact and there exists a real-value polynomial $u(z)$ such that $\{z : u(z) \geq 0\}$ is compact and:

$$u(z) = u_0(z) + \sum_{i=1}^{n_q} q_i(z) u_i(z), \tag{61}$$

where $u_i(z)$ (with $i = 0, \ldots, n_q$) are all sum-of-squares polynomials. Then, any polynomial $t(z)$ strictly positive on $\boldsymbol{Z}$ can be written as:

$$t(z) = \sigma_0(z) + \sum_{i=1}^{n_q} q_i(z) \sigma_i(z),$$

where $\sigma_i(z)$ (with $i = 0, \ldots, n_q$) are all sum-of-squares polynomials (whose degree is not known in advance).

Note that if the set $\boldsymbol{Z}$ is included in the ball $\{z : \|z\|^2 \leq a^2\}$, for a sufficiently large, one way to ensure that the assumptions in Lemma 1 are satisfied is to add in the definition of $\boldsymbol{Z}$ the constraint $q_{n_q+1}(z) = a^2 - \|z\|^2 \geq 0$ and chose in (61) $u_i = 0$ $(i = 0, \ldots, n_q)$ and $u_{n_q+1} = 1$.

Proposition 1 *The dual of the semi-infinite optimization problem* (30) *is:*

$$t^* = \inf_{\nu} \;\; \nu_1 + \sum_{i=2}^{n_f} \nu_i \mu_i \tag{62a}$$

$$s.t. \;\; \nu_1 + \sum_{i=2}^{n_f} \nu_i \tilde{f}_i(z) - h(z) \geq 0 \;\;\; \forall z \in \boldsymbol{Z}. \tag{62b}$$

From well known results on dual optimization [47], if μ belongs to the interior of the moment space generated by $P_z \in \mathcal{P}_z(\mu)$, then there is no duality gap between problem (30) *and problem* (62)*, i.e.,*

$$t^* = \overline{p}. \tag{63}$$

Proposition 2 *Let us write the moment matrix $M_\tau(m)$ and the localizing matrices $M_{\tau - \left\lceil \frac{deg(q_j)}{2} \right\rceil}(q_j m)$ in* (43) *as*

$$M_\tau(m) = \sum_{\alpha \in \mathcal{A}_{2\tau}^{n_z}} B_\alpha m_\alpha, \tag{64a}$$

$$M_{\tau - \left\lceil \frac{deg(q_j)}{2} \right\rceil}(q_j m) = \sum_{\alpha \in \mathcal{A}_{2\tau}^{n_z}} C_\alpha^{(j)} m_\alpha, \tag{64b}$$

where B_α and $C_\alpha^{(j)}$ are symmetric matrices properly defined.

Then, the dual of the SDP problem (43) *is given by:*

$$\overline{t}^\tau = \inf_{\nu, X \succeq 0, Y^{(j)} \succeq 0} \quad \nu^\top \mu \tag{65a}$$

$$s.t.\ \nu_1 + \sum_{i=2}^{n_f} \nu_i \tilde{\boldsymbol{f}}_{i,\alpha} - \boldsymbol{h}_\alpha = <X, B_\alpha> + \sum_{j=1}^{n_q} <Y^{(j)}, C_\alpha^{(j)}>, \alpha \in \mathcal{A}_{2\tau}^{n_z}, \tag{65b}$$

with $<X, B_\alpha>$ (resp. $<Y^{(j)}, C_\alpha^{(j)}>$) being the trace of the matrix $X B_\alpha$ (resp. $Y^{(j)} C_\alpha^{(j)}$).

Obviously, by weak duality:

$$\overline{t}^\tau \geq \overline{p}^\tau. \tag{66}$$

Theorem 5 *Under the assumptions in Lemma 1 and Proposition 1, the following convergence condition holds:* $\lim_{\tau \to \infty} \overline{p}^\tau = \overline{p}$.

Proof et ν^* be the optimal solution of problem (62). Thus: $t^* = \nu_1^* + \sum_{i=2}^{n_f} \nu_i^* \mu_i$, and $\nu_1^* + \sum_{i=2}^{n_f} \nu_i^* \tilde{f}_i(z) - h(z) \geq 0 \quad \forall z \in \boldsymbol{Z}$. Take $\varepsilon > 0$ arbitrary. Then:

$$\nu_1^* + \sum_{i=2}^{n_f} \nu_i^* \tilde{f}_i(z) - h(z) + \varepsilon > 0 \quad \forall z \in \boldsymbol{Z}.$$

Since the polynomial $\nu_1^* + \sum_{i=2}^{n_f} \nu_i^* \tilde{f}_i(z) - h(z) + \varepsilon$ is strictly positive on $\boldsymbol{Z}$, because of Lemma 1, there exist sum-of-squares polynomials $\sigma_j(z)$ ($j = 0, \ldots, n_q$) such that

$$\nu_1^* + \sum_{i=2}^{n_f} \nu_i^* \tilde{f}_i(z) - h(z) + \varepsilon = \sigma_0(z) + \sum_{j=1}^{n_q} q_j(z) \sigma_j(z),$$

provided that $\sigma_0(z)$ and $\sigma_j(z)$ ($j = 1, \ldots, n_q$) have order 2τ and $2\tau - 2\left\lceil \frac{deg(q_j)}{2} \right\rceil$, respectively, for τ large enough.

Let us write the SOS polynomials $\sigma_j(z)$ ($j = 0, \ldots, n_q$) as $\sigma_j(z) = \sum_{i=1}^{r_j} \sigma_{ji}(z)^2$, and let $\boldsymbol{\sigma}_{ji}$ be the vector of coefficients of the polynomial $\sigma_{ji}(z) \in \mathbb{R}_{d_j}[z]$ in the basis $b_{d_j}(z)$, with $d_0 = \tau$, $d_j = \tau - \left\lceil \frac{deg(q_j)}{2} \right\rceil$, $j = 1, \ldots, n_q$. Let us construct the matrices

$$X = \sum_{i=1}^{r_0} \boldsymbol{\sigma}_{0i} \boldsymbol{\sigma}'_{0i} \succeq 0, \quad Y^{(j)} = \sum_{i=1}^{r_j} \boldsymbol{\sigma}_{ji} \boldsymbol{\sigma}'_{ji} \succeq 0. \tag{67}$$

For an arbitrary $z \in \mathbb{R}^{n_z}$, let us construct the vector

$$m = b_{2\tau}(z) = \begin{bmatrix} 1 & z_1 & \cdots & z_{n_z} & z_1^2 & z_1 z_2 & \cdots & z_{n_z}^{2\tau} \end{bmatrix}. \tag{68}$$

Then, with m as in (68), and X and $Y^{(j)}$ as in (67), we have

$$\begin{aligned} & < X, M_\tau(m) > + \sum_{j=1}^{n_q} < Y^{(j)}, M_{\tau - \left\lceil \frac{deg(q_j)}{2} \right\rceil}(q_j m) > \\ = & \sigma_0(z) + \sum_{j=1}^{n_q} q_j(z) \sigma_j(z) = \nu_1^* + \sum_{i=2}^{n_f} \nu_i^* \tilde{f}_i(z) - h(z) + \varepsilon. \end{aligned} \tag{69}$$

Since z in (68) is arbitrary, condition (69) holds for any $z \in \mathbb{R}^{n_z}$. Thus, by rewriting $M_\tau(m)$ and $M_{\tau - \left\lceil \frac{deg(q_j)}{2} \right\rceil}(q_j m)$ as in (64), we have:

$$<X, B_\alpha> + \sum_{j=1}^{n_q} <Y^{(j)}, C_\alpha^{(j)}> = \nu_1^* + \sum_{i=2}^{n_f} \nu_i^* \tilde{\boldsymbol{f}}_{i,\alpha} - \boldsymbol{h}_\alpha + \varepsilon, \ \alpha \in \mathcal{A}_{2\tau}^{n_z}.$$

Thus, $\nu_1 = \nu_1^* + \varepsilon$, $\nu_i = \nu_i^*$ $(i = 2, \ldots, n_f)$, and X and $Y^{(j)}$ in (67) are feasible for problem (65). For these values of ν, X and $Y^{(j)}$, the cost function in (67) is equal to $\nu_1^* + \sum_{i=2}^{n_f} \nu_i^* \mu_i + \varepsilon = t^* + \varepsilon$. Therefore,

$$\bar{t}^\tau \leq t^* + \varepsilon. \tag{70}$$

By combining eqs. (45), (63), (66) and (70), we have:

$$t^* = \overline{p} \leq \overline{p}^\tau \leq \bar{t}^\tau \leq t^* + \varepsilon. \tag{71}$$

Summarizing, for every $\varepsilon > 0$, there exists τ large enough such that (see (71)): $\overline{p} \leq \overline{p}^\tau \leq \overline{p} + \varepsilon$, or equivalently, $\lim_{\tau \to \infty} \overline{p}^\tau = \overline{p}$.

Acknowledgment

The authors would like to thank Prof. Nicola Guglielmi for the interesting discussions on $\mathcal{D}$-stability analysis and Prof. Johan Löfberg for his suggestions on the implementation of the Lasserre's hierarchy with moment constraints in YALMIP.

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