

Quantum Bounded Query Complexity

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Abstract

We combine the classical notions and techniques for bounded query classes with those developed in quantum computing. We give strong evidence that quantum queries to an oracle in the class NP does indeed reduce the query complexity of decision problems. Under traditional complexity assumptions, we obtain an exponential speedup between the quantum and the classical query complexity of function classes.

For decision problems and function classes we obtain the following results:

- $P_{||}^{\text{NP}[2k]} \subseteq \text{EQP}_{||}^{\text{NP}[k]}$
- $P_{||}^{\text{NP}[2^{k+1}-2]} \subseteq \text{EQP}_{||}^{\text{NP}[k]}$
- $\text{FP}_{||}^{\text{NP}[2^{k+1}-2]} \subseteq \text{FEQP}_{||}^{\text{NP}[2k]}$
- $\text{FP}_{||}^{\text{NP}} \subseteq \text{FEQP}_{||}^{\text{NP}[O(\log n)]}$

For sets A that are many-one complete for PSPACE or EXP we show that $\text{FP}^A \subseteq \text{FEQP}^{A[1]}$. Sets A that are many-one complete for PP have the property that $\text{FP}_{||}^A \subseteq \text{FEQP}^{A[1]}$. In general we prove that for any set A there is a set X such that $\text{FP}^A \subseteq \text{FEQP}^{X[1]}$, establishing that no set is superterse in the quantum setting.

1 Introduction

The query complexity of a function is the minimum number of queries (to some oracle) that are needed to compute one value of this function. With *bounded* query complexity we look at the set of functions that can be calculated if we put an upper bound on the number of queries that we allow the computer to ask the oracle. This notion has been extensively studied both in the resource bounded setting [1, 3, 4, 7, 8, 9, 13, 31, 33, 36] and in the recursive setting [11, 12]. This notion and its variants has lead to a series of techniques and tools that are used throughout complexity theory.

In this paper we combine some of the bounded query notions with quantum computation. The main goal of the paper is to further—as was done by Fortnow and Rogers [27]—the incorporation of quantum computation into complexity theory. We feel that the synthesis of quantum computation

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and classical complexity theory serves two purposes. First, it is important to know the limits of feasible quantum computation and this can be done by incorporating it into the framework of classical computation. Second, the insights of quantum computation can be useful for classical complexity theory in turn.

We start out with the class of sets (or decision problems) that are computable in polynomial time with bounded queries to a set in NP. We consider the setting where the queries are adaptive (i.e., a query may depend on the answers to previous ones), as well as where they are non-adaptive. Classically, it is known that any decision problem that can be solved in polynomial time with k adaptive queries to a set in NP (the class $P^{\text{NP}[k]}$) can also be solved with $2^k - 1$ non-adaptive queries (the class $P_{\parallel}^{\text{NP}[2^k-1]}$, where “ \parallel ” indicates the parallel or non-adaptive queries), and vice-versa [9]. In other words: $P^{\text{NP}[k]} = P_{\parallel}^{\text{NP}[2^k-1]}$. Moreover, there is strong evidence that this trade-off is optimal in the sense that every non-adaptive class $P_{\parallel}^{\text{NP}[k]}$ is different for different values of k . For example if $P_{\parallel}^{\text{NP}[2]} \subseteq P^{\text{NP}[1]}$, then the polynomial hierarchy collapses [31] (see also [17, 29]).

The natural quantum analogue of P is the class EQP, which stands for *exact quantum polynomial time*. This is the class of sets or decision problems that is computable in polynomial time with a quantum computer that makes no errors (i.e., is exact). In this paper we will see that if we allow the query machine to make use of quantum mechanical effects such as superposition and interference the situation changes. In the non-adaptive case we will show that $2k$ classical queries can be simulated with only k non-adaptive ones on a quantum computer and in the adaptive case we show how to simulate $2^{k+1} - 2$ classical queries with only k quantum queries. Hence

$$P_{\parallel}^{\text{NP}[2k]} \subseteq \text{EQP}_{\parallel}^{\text{NP}[k]} \quad \text{and} \quad P_{\parallel}^{\text{NP}[2^{k+1}-2]} \subseteq \text{EQP}^{\text{NP}[k]}.$$

In particular it follows from this result that $P_{\parallel}^{\text{NP}[2]} \subseteq \text{EQP}^{\text{NP}[1]}$ (see also [22]).

In order to prove these results we combine the classical mind-change technique [9] with the one query version (see [20]) of the first quantum algorithm developed by David Deutsch [23].

Next, we turn our attention to *functions* that are computable with bounded queries to a set in NP. Compared to the decision problems there is probably no nice trade-off between adaptive and non-adaptive queries for functions. This is because the following is known [13]: for any k the inclusion $\text{FP}_{\parallel}^{\text{NP}[k]} \subseteq \text{FP}^{\text{NP}[k-1]}$ implies that $P = \text{NP}$. Moreover, if $\text{FP}_{\parallel}^{\text{NP}} \subseteq \text{FP}^{\text{NP}[O(\log n)]}$ then the polynomial time hierarchy collapses [8, 34, 35].

When the adaptive query machine is a quantum computer, things are different and we seem to get a trade-off between adaptiveness and query complexity. We show the following:

$$\text{FP}_{\parallel}^{\text{NP}[2^{k+1}-2]} \subseteq \text{FEQP}^{\text{NP}[2k]} \quad \text{and} \quad \text{FP}_{\parallel}^{\text{NP}} \subseteq \text{FEQP}^{\text{NP}[O(\log n)]}.$$

Here $\text{FEQP}^{\text{NP}[k]}$ is the class of functions that is computable by an exact quantum Turing machine that runs in polynomial time and is allowed to make k queries to a set in NP. The proofs of these results use our previous results on decision problems and a quantum algorithm developed by Deutsch-Jozsa [24] and Bernstein-Vazirani [15].

Using the same ideas we are able to show that for any set A there exists a set X such that $\text{FP}^A \subseteq \text{FEQP}^{X[1]}$, establishing that no set is ‘superterse’. Also because the complexity of X is not much harder than that of A (the problem X is Turing reducible to A), we get quite general theorems for complete sets of complexity classes.

For a complexity class \mathcal{C} that is closed under Turing reductions, and a problem $A \in \mathcal{C}$ that is many-one complete for the class \mathcal{C} , the inclusion $\text{FP}^{\mathcal{C}} \subseteq \text{FEQP}^{A[1]}$ is proven. This holds in particular for the set QBF of the *true quantified Boolean formulae* which is a PSPACE complete problem, and the complete sets for the class EXP. If \mathcal{C} is a class that is closed under truth-table reductions, then it holds that $\text{FP}_{\parallel}^{\mathcal{C}} \subseteq \text{FEQP}^{A[1]}$. The Theta levels of the polynomial hierarchy and PP are examples of such classes.

The ingredients for all our results are standard quantum algorithms combined with well known techniques from complexity theory. Nevertheless we feel that this combination gives a new point of view on the nature of bounded query classes and the structure of complete sets in general.

2 Preliminaries

2.1 Classical computing

We assume the reader to be familiar with basic notions of complexity theory such as the various complexity classes and types of reducibility as can be found in many textbooks in the area [5, 6, 28, 30]. The essentials for this article are mentioned below.

For a set (decision problem) A we will identify A with its characteristic function. Hence for a string x we have $A(x) \in \{0, 1\}$, and $A(x) = 1$ if and only if $x \in A$. A class \mathcal{C} consists of a set of decision problems. A problem A is many-one, or \leq_m^p -complete for a class \mathcal{C} if for any problem $B \in \mathcal{C}$, there exists a polynomial function or “Karp-reduction” τ such that $x \in B$ if and only if $\tau(x) \in A$. The typical example of such a complete problem is SAT (the set of satisfiable Boolean formulae) which is \leq_m^p -complete for the class NP. The class FP indicates the set of *functions* that can be calculated on a polynomial time, deterministic Turing machine.

An oracle Turing machine is *non-adaptive*, if it can produce a list of all of the oracle queries it is going to make before it makes the first query. For any set A , the elements of the class $P^{A[k]}$ ($FP^{A[k]}$) are the languages (functions) that are computable by polynomial time Turing machines that accesses the oracle A at most k times on each input. The class $P_{||}^{A[k]}$ and $FP_{||}^{A[k]}$ allow only non-adaptive access to A . The notation $P^{NP[q(n)]}$ is used to indicate algorithms that might require $q(n)$ oracle calls, where q is a function of the input size n .

The class NP can be generalised by defining the *polynomial time hierarchy*. We start with the definition $\Sigma_0^p = P$ and then for the higher levels continue in an inductive fashion with $\Sigma_{i+1}^p = NP^{\Sigma_i^p}$ for $i = 1, 2, \dots$. Many complexity theorists conjecture that this polynomial time hierarchy is infinite, i.e., $\Sigma_{i+1}^p \neq \Sigma_i^p$ for all i .

A class \mathcal{C} of languages is closed under Turing (truth-table) reduction if any decision problem that can be solved with a polynomial time Turing machine and (non-adaptive) queries to a set in \mathcal{C} , is itself also an element of \mathcal{C} . Examples of such classes are PSPACE, EXP, and the Delta levels $\Delta_{i+1}^p = P^{\Sigma_i^p}$ of the polynomial time hierarchy. The classes PP and $\Theta_{i+1}^p = P_{||}^{\Sigma_i^p}$ (Theta levels of the polynomial hierarchy) are for example closed under this truth-table-reduction.

2.2 Quantum computing

In this section we define quantum oracle Turing Machines. For an introduction to quantum computing see for example the survey by Berthiaume [16].

A *qubit* is a superposition $\alpha_0|0\rangle + \alpha_1|1\rangle$ of both values of a classical bit. The complex values α_0 and α_1 are the *amplitudes* of the quantum state, and they obey the normalisation restriction: $|\alpha_0|^2 + |\alpha_1|^2 = 1$.

The tensor or Kronecker-product is used to describe a system of several qubits. The combination of two qubits is thus calculated by

$$\begin{aligned} |x\rangle \otimes |y\rangle &= (\alpha_0|0\rangle + \alpha_1|1\rangle) \otimes (\beta_0|0\rangle + \beta_1|1\rangle) \\ &= \alpha_0\beta_0|00\rangle + \alpha_0\beta_1|01\rangle + \alpha_1\beta_0|10\rangle + \alpha_1\beta_1|11\rangle. \end{aligned}$$

Consequently, a register of n qubits is a superposition $|\psi\rangle$ of all 2^n strings of n classical bits, written

$$|\psi\rangle = \sum_{i \in \{0,1\}^n} \alpha_i |i\rangle.$$

If we measure the quantum register $|\psi\rangle$ in the standard (i.e. classical) basis, we will observe one and only one of the basis states $|i\rangle$ with probability $|\alpha_i|^2$. Hence we must have $\sum_{i \in \{0,1\}^n} |\alpha_i|^2 = 1$ (the normalisation restriction). After measuring $|\psi\rangle$ and observing $|i\rangle$, we say that the superposition $|\psi\rangle$ “has collapsed” to the new state $|i\rangle$.

If we do not observe a state, quantum mechanics tells us that it will evolve *unitarily*, as this is the only evolution that respects the normalisation restriction. Unitarity means that the vector of

amplitudes is transformed according to a linear operator that preserves the unit norm. This can be viewed as a rotation in the complex, finite Hilbert space of dimension 2^n . A unitary operator U always has an inverse U^{-1} which equals its conjugate transpose U^\dagger .

An example of a one-qubit operation is the Hadamard transform H which is defined by

$$\begin{aligned} H|0\rangle &= \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \\ H|1\rangle &= \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle). \end{aligned}$$

The effect of the n -fold tensor product of $H^{\otimes n}$ on n qubits is now calculated by the linearity of quantum mechanics:

$$H^{\otimes n} \sum_{i \in \{0,1\}^n} \alpha_i |i\rangle = \sum_{i \in \{0,1\}^n} \alpha_i \cdot H^{\otimes n} |i\rangle.$$

This shows that the evolution of the superposition $|\psi\rangle$ is fully determined by the evolution of its basis states $|i\rangle$. We will use this transformation $H^{\otimes n}$ in Section 4.

A quantum Turing machine's transition function can be described by a unitary matrix M with complex entries. The computation of t time-steps will then correspond to the application of the matrix product M^t to the initial configuration $|j\rangle$. At the end of the computation we measure the state that has evolved in this manner and accept j if some designated bit is 1 and otherwise reject the input j .

The class EQP is defined as those sets that can be computed by a quantum Turing machine that runs in polynomial time and accepts every string j with probability 1 or 0. Likewise, we define the class of functions FEQP as the class of functions that can be computed by some quantum Turing machine that runs in polynomial time. The output of the Turing machine may now be several bits.

We model oracle computation as follows (see also [14]). An oracle Turing machine has some special query tape, and at some point in the computation the Turing machine may go into a special pre-query state to make a query to the oracle set A . Suppose the query tape contains the state $|i\rangle|b\rangle$ (i represents the query and b is a bit). The result of this operation is that after the call the machine will go into a special state called the post-query state and that the query tape has changed into $|i\rangle|A(i) \oplus b\rangle$, where \oplus is the EXCLUSIVE OR. We will denote this unitary operation by U_A . Note that U_A only changes the contents of the special query tape b , and leaves all the other registers unchanged.

As with classical oracle computation, we make the distinction between adaptive and non-adaptive quantum oracle machines. We call a quantum oracle machine non-adaptive if on every computation path a list of all the oracle queries (on this path) is generated before the first query is made.

The class EQP $^{A[k]}$ are the sets recognised by an exact quantum Turing machine that runs in polynomial time and makes at most k queries to the oracle for A . Again, we define classes like EQP $_{||}^{A[q(n)]}$, FEQP $^{A[q(n)]}$, and FEQP $_{||}^{A[q(n)]}$, for non-adaptive decision, adaptive function, and non-adaptive function classes respectively (with $q(n)$ a function that gives an upper bound on the number of queries and n the size of the input string).

3 Decision Problems

In this section we will investigate the extra power that a polynomial time, exact quantum computer yields compared to classical deterministic computation when querying a set in the class NP. In the case of deterministic computation the following equality between adaptive and non-adaptive queries to NP is well known.

Theorem 1 [9, 19, 36]

1. For any $k \geq 0$ we have $P_{||}^{\text{NP}[2^k-1]} = P^{\text{NP}[k]}$.
2. For any polynomial $q(n) > 1$ the equality $P_{||}^{\text{NP}[q(n)]} = P^{\text{NP}[O(\log(q(n)))]}$ holds.

Proof: Both items are proved in a similar way which has two parts. The first part shows that computing a function in $P_{||}^{\text{NP}[2^k-1]}$ can be reduced to computing the *parity* of $2^k - 1$ other queries to NP. The second part then proceeds by showing that using binary search one can compute the parity of $2^k - 1$ NP-queries with k *adaptive* queries to SAT. On the other hand, it is trivial to see that any computation with k adaptive queries can be simulated exhaustively with $2^k - 1$ non-adaptive oracle calls. \square

There is also strong evidence that the above trade-off is tight (see [10, 31]). It follows for example that if $P_{||}^{\text{NP}[2]} = P^{\text{NP}[1]}$ then the polynomial hierarchy collapses [31]. (See [17] for the latest developments with respect to this question.)

Perhaps surprisingly the situation changes when the query machine is quantum mechanical. David Deutsch [23] developed a quantum algorithm to compute the parity of any two Boolean variables in *one* query with higher probability than a classical (randomised) algorithm can. Cleve *et al.* [20] showed how to make this procedure exact.

Theorem 2 [20, 23] *Let $f : \{0, 1\} \mapsto \{0, 1\}$. There exists an exact quantum algorithm that computes the parity bit $f(0) \oplus f(1)$ with one query to the function f . This algorithm works in constant time.*

Proof: For simplicity we only describe what is happening to the states that get effected by the oracle query. Construct the following initial state:

$$|\text{Initial}\rangle = \frac{1}{2}(|0\rangle + |1\rangle) \otimes (|0\rangle - |1\rangle). \quad (1)$$

Next, make the only query to f depending on the value of the first bit. Note that f will thus be queried in superposition for both $f(0)$ and $f(1)$. Applying f establishes the following evolution on the two qubits:

$$|i\rangle \otimes |b\rangle \longrightarrow |i\rangle \otimes |b \oplus f(i)\rangle.$$

This results in the following outcome when applied to the initial state:

$$\begin{aligned} & \frac{(-1)^{f(0)}}{2}(|0\rangle + |1\rangle) \otimes (|0\rangle - |1\rangle) \quad \text{if } f(0) = f(1) \\ & \frac{(-1)^{f(0)}}{2}(|0\rangle - |1\rangle) \otimes (|0\rangle - |1\rangle) \quad \text{if } f(0) \neq f(1). \end{aligned}$$

Which means that if we apply a Hadamard transformation to the first register, we obtain

$$|\text{Final}\rangle = (-1)^{f(0)}|f(0) \oplus f(1)\rangle \otimes (|0\rangle - |1\rangle).$$

Hence observing the first bit yields the correct answer $f(0) \oplus f(1)$. \square

Using this procedure we will now show that a quantum Turing machine can compute decision problems with half the number of non-adaptive queries.

Theorem 3 *For any $k \geq 0$ we have the inclusion $P_{||}^{\text{NP}[2k]} \subseteq \text{EQP}_{||}^{\text{NP}[k]}$.*

Proof: Without loss of generality we will assume that the queries are made to SAT, and that the predicate that is computable with $2k$ queries to SAT is $f(x)$. Let $\psi_1, \psi_2, \dots, \psi_{2k}$ be the queries that the computation of $f(x)$ makes. We will use the proof technique of Theorem 1 (also

called mind-change technique) which enables us to compute $f(x)$ by calculating the single bit $\text{SAT}(\phi_1) \oplus \dots \oplus \text{SAT}(\phi_{2k})$. Here the new formulae ϕ_1, \dots, ϕ_{2k} can be computed in polynomial time from $\psi_1, \dots, \psi_{2k}, f$, and x , but without having to consult SAT.

Next, we use Theorem 2 to compute the parity $\text{SAT}(\phi_i) \oplus \text{SAT}(\phi_{i+1})$ for odd i ($1 \leq i < 2k$) with k non-adaptive queries to SAT. Finally we compute the parity of these answers, thus obtaining the necessary information for calculating $f(x)$. \square

Corollary 1 $\text{P}_{||}^{\text{NP}[2]} \subseteq \text{EQP}^{\text{NP}[1]}$ (see [22]).

We do not know whether this is tight. It would be interesting to either improve this result to $\text{P}^{\text{NP}[2]} \subseteq \text{EQP}^{\text{NP}[1]}$ or to show as a consequence of this that the polynomial time hierarchy collapses.

Theorem 1 relates adaptive query classes to non-adaptive ones, thereby establishing an exponential gain in the number of queries ($2^k - 1$ versus k queries). We will now show how to use the Deutsch trick to do even slightly better than that in the quantum case.

Theorem 4 $\text{P}_{||}^{\text{NP}[2^{k+1}-2]} \subseteq \text{EQP}^{\text{NP}[k]}$ for any $k \geq 0$.

Proof: The proof is by induction on k . For $k = 1$ we have back the situation of Corollary 1. Let the predicate $f(x)$ be computable with $2^{k+1} - 2$ non-adaptive queries to SAT. As in the proof of Theorem 3 we reduce the $2^{k+1} - 2$ queries ψ_i that $f(x)$ makes, to the calculation of the parity-bit $\text{SAT}(\phi_1) \oplus \dots \oplus \text{SAT}(\phi_{2^{k+1}-2})$. Next, we construct $2^{k+1} - 2$ new formulae $\chi_1, \dots, \chi_{2^{k+1}-2}$ according to:

$$\chi_i \text{ is satisfiable} \iff |\{\phi_1, \dots, \phi_{2^{k+1}-2}\} \cap \text{SAT}| \geq i.$$

The construction of each such χ_i can be done in polynomial time. To see this, consider the non-deterministic polynomial time Turing machine M that on input $\langle i, \phi_1, \dots, \phi_{2^{k+1}-2} \rangle$, accepts if and only if it can find for i of the formulae a satisfying assignment. Cook and Levin [21, 32] —proving that SAT is \leq_m^p -complete for NP— showed that any polynomial time non-deterministic Turing machine computation $M(x)$ in polynomial time can be transformed into a formula that is satisfiable if and only if $M(x)$ has an accepting computation. Let χ_i be the result of this Cook-Levin reduction.

Note the following two properties of those formulae χ_i :

1. The parity $\text{SAT}(\phi_1) \oplus \dots \oplus \text{SAT}(\phi_{2^{k+1}-2})$ is the same as the parity $\text{SAT}(\chi_1) \oplus \dots \oplus \text{SAT}(\chi_{2^{k+1}-2})$.
2. For every i we have $\text{SAT}(\chi_i) \geq \text{SAT}(\chi_{i+1})$.

Now we are ready to make the first query. We compute the parity of $\chi_{2^{k-1}}$ and $\chi_{2^{k-1}+2^k-1}$. This can be done in one query using Theorem 2. By doing this we have at the cost of one query reduced the question of computing the parity of $2^{k+1} - 2$ formulae to computing the parity of $2^k - 2$. These we can solve using $k - 1$ queries using the induction hypothesis. To see this observe the following. For convenience set $a = 2^{k-1}$ and $b = 2^{k-1} + 2^k - 1$.

Suppose the parity of χ_a and χ_b is odd. Hence $\chi_1, \chi_2, \dots, \chi_a$ are all satisfiable and $\chi_b, \dots, \chi_{2^{k+1}-2}$ all un-satisfiable (using property 2 above). Also note that a is even, so the parity of $\chi_1, \dots, \chi_{2^{k+1}-2}$ is the same as the parity of $\chi_{a+1}, \dots, \chi_{b-1}$ (these are $2^k - 2$ many formulae).

On the other hand assume that the parity of χ_a and χ_b is even. This means (again using property 2 above) that χ_a, \dots, χ_b are all either satisfiable or un-satisfiable and hence have even parity. So again the question reduces to the parity of the remaining formulae: $\chi_1, \dots, \chi_{a-1}$ and $\chi_{b+1}, \dots, \chi_{2^{k+1}-2}$. Which happen to be $2^k - 2$ many formulae. \square

We do not know if it is possible to do better than this. In essence the above technique seems to boil down to searching in an ordered list. Buhrman and De Wolf [18] show that this can not be done faster than $\sqrt{\log n} / \log(\log n)$, which was improved by Fahri *et al.* [25] to $\log(n) / 2 \log(\log(n))$ and later by Ambainis [2] to a lower bound of $1/12 \log n - O(1)$ queries. Recent results by Farhi *et al.* [26] seem to suggest a reduction in the query complexity by a factor of two. But it is not clear if their exact quantum algorithm for ‘insertion into an ordered list’ translates correctly into our setting.

4 Function Classes

Now we turn our attention to function classes where the algorithm can output bit *strings* rather than single bits. We will see that in this scenario the difference between classical and quantum computation becomes more eminent.

4.1 Functions computable with queries to an oracle in NP

We start out by looking at functions that are computable with queries to a complete set for the class NP. Classically the situation is not as well understood as the class of decision problems. There is strong evidence that the analogue of Theorem 1 is not true.

Theorem 5 *The following holds for the classical, exact computation of functions:*

1. *If for some $k \geq 0$ we have $\text{FP}_{\parallel}^{\text{NP}[k+1]} \subseteq \text{FP}_{\parallel}^{\text{NP}[k]}$, then $\text{P} = \text{NP}$ [13].*
2. *If for all polynomials $q(n)$ (with n the size of the input string): $\text{FP}_{\parallel}^{\text{NP}[q(n)]} \subseteq \text{FP}_{\parallel}^{\text{NP}[O(\log n)]}$, then $\text{NP} = \text{R}$ (and the polynomial hierarchy collapses) [8, 34, 35].*

When we allow the adaptive query machine to be quantum mechanical the picture becomes again quite different. We will show for example that the inclusion $\text{FP}_{\parallel}^{\text{NP}[q(n)]} \subseteq \text{FEQP}_{\parallel}^{\text{NP}[2\log(q(n))]}$ holds. In order to do so we will first need a generalisation of the parity trick of the Theorem 2.

Deutsch and Jozsa [24] generalised the setting to that of Boolean functions $f : \{0, 1\}^n \mapsto \{0, 1\}$ with the promise that f has either an equal number of 0 and 1 outputs—they called such functions *balanced*—or f is *constant* on all inputs (0 or 1). Classically one needs $2^n/2 + 1$ applications of f to determine whether it is constant or balanced. Deutsch and Jozsa however showed that this can be done with 2 applications in the quantum setting. Cleve *et al.* [20] demonstrated how to do this with a single quantum query. Based upon the work of Deutsch and Jozsa, Bernstein and Vazirani [15] considered a subclass of the constant and balanced functions. Suppose that $f : \{0, 1\}^n \mapsto \{0, 1\}$ is such that:

$$f(x) = (a_1 \wedge x_1) \oplus \cdots \oplus (a_n \wedge x_n) = (a, x). \quad (2)$$

Where $(a_1 a_2 \cdots a_n) = a \in \{0, 1\}^n$, and (a, x) is the inner product of the vectors a and x modulo 2. The goal is, given f , to determine a . Bernstein and Vazirani showed that this can be done with 2 applications of f on a quantum computer. Classically one needs at least n applications. Cleve *et al.* [20] improved this again to only 1 application of the function f .

The core of the algorithm is the following observation. The n fold Hadamard transform $H^{\otimes n}$ (see Section 2.2) does the following when applied to a basis state of n bits:

$$H^{\otimes n} |a_1 a_2 \cdots a_n\rangle = \frac{1}{\sqrt{2^n}} \sum_{x \in \{0, 1\}^n} (-1)^{(x, a)} |x\rangle. \quad (3)$$

Since the Hadamard transform is its own inverse we have also the other direction:

$$H^{\otimes n} \frac{1}{\sqrt{2^n}} \sum_{x \in \{0, 1\}^n} (-1)^{(x, a)} |x\rangle = |a_1 a_2 \cdots a_n\rangle. \quad (4)$$

So, if we are able to obtain the state of Equation 3, then we can *extract* the n -bit string a out of it by simply applying $H^{\otimes n}$ to it. This state however can be obtained with one application to f as follows:

$$U_f \frac{1}{\sqrt{2^{n+1}}} \sum_{x \in \{0, 1\}^n} |x\rangle(|0\rangle - |1\rangle) = \frac{1}{\sqrt{2^{n+1}}} \sum_{x \in \{0, 1\}^n} (-1)^{f(x)} |x\rangle(|0\rangle - |1\rangle). \quad (5)$$

Now observe that the last qubit is always in state $(|0\rangle - |1\rangle)$. Using the definition of f we can rewrite this state to:

$$\frac{1}{\sqrt{2^{n+1}}} \sum_{x \in \{0,1\}^n} (-1)^{(x,a)} |x\rangle \otimes (|0\rangle - |1\rangle).$$

Let us turn back now to our setting of bounded query classes. Using the above quantum tricks we can show the following.

Theorem 6 *For exact function calculation with the use of an oracle in NP it holds that*

1. $\text{FP}_{\parallel}^{\text{NP}[2^{k+1}-2]} \subseteq \text{FEQP}^{\text{NP}[2k]}$ for any $k \geq 0$.
2. $\text{FP}_{\parallel}^{\text{NP}} \subseteq \text{FEQP}^{\text{NP}[O(\log n)]}$.

Proof: Fix $k \geq 0$, the input z of length m and let g be the function in $\text{FP}_{\parallel}^{\text{SAT}[2^{k+1}-2]}$. Suppose that $g(z) = (a_1 \cdots a_n) = a$ with $n = m^c$ for some c depending on g . The goal is to obtain the following state:

$$|\text{Output}\rangle = \frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} (-1)^{(x,a)} |x\rangle. \quad (6)$$

Since with this state one application of $H^{\otimes n}$ will give us $a = g(z)$ (see Equation 4). Similar to Equation 5 we can obtain this state if we had access to a function f with the same property as the one in Equation 2.

The goal thus is to transform the function we *have* access to—SAT in our case—into one that resembles the one in Equation 2. The way to do this is to make use of a quantum subroutine. Observe the following: the binary function $f_z(x) = (x, a)$ is in $\text{P}_{\parallel}^{\text{SAT}[2^{k+1}-2]}$ because we can first compute $g(z) = a$ with $2^{k+1} - 2$ queries to SAT and then determine (x, a) . By Theorem 4 this function is computable in $\text{EQP}^{\text{SAT}[k]}$. Hence, when we use this adaptive EQP algorithm in superposition we have the desired function f . There is however one problem with this approach. The algorithm that comes out of Theorem 4 leaves several of the registers in states depending on the input x and SAT. For example the algorithm that computes the parity of two function calls in one generates a phase of (-1) depending on the value of the first function call (see Equation 2). These changes in registers and phase shifts obstruct our base quantum machine and as a consequence the sum computed in Equation 4 does not work out the way we want (i.e., the interference pattern is different and terms do not cancel out as nice as before.)

The solution to this kind of ‘garbage’ problem is as follows:

1. Compute $f_z(x)$ with k queries to SAT.
2. Copy the outcome onto an extra auxiliary qubit (by setting the auxiliary bit b to the EXCLUSIVE OR of b and the outcome).
3. Reverse the computation of $f_z(x)$ making another k queries to SAT.

Observe that when we compute $f_z(x)$ in this way, all the phase changes and registers are reset and are in the same state as before computing f , except for the auxiliary qubit that contains the answer. Since the subroutine was exact (i.e., in EQP) the answer bit is a classical bit and will not interfere with the rest of the computation. Note (see Section 2) that this corresponds exactly to one oracle call to f . Thus we simulated 1 call to f with $2k$ queries to SAT and hence have established a way of producing the desired state of Equation 6.

The second part of the theorem is proved in a similar way now using part 2 of Theorem 1. \square

4.2 Terseness, and other complexity classes

The quantum techniques described above are quite general and can be applied to sets outside of NP. Classically the following question has been studied (see [8] for more information). For any set A define the function $F_n^A(x_1, \dots, x_n) = (A(x_1) \cdots A(x_n))$ which is an n bit vector telling which of the x_i 's is in A and which ones are not. A basic question now is: how many queries to A do we need to compute F_n^A ? Sets for which F_n^A can not be computed with less than n queries to A (i.e., $F_n^A \not\subseteq \text{FP}^{A[n-1]}$) are called P -terse. We call the decision problem A P -superterse if $F_n^A \not\subseteq \text{FP}^{X[n-1]}$ for any set X . The next theorem shows that the notion P -superterse is not useful in the quantum setting.

Theorem 7 *For any set A there exists a set X such that for all n we have $F_n^A \subseteq \text{FEQP}^{X[1]}$.*

Proof: Let X be the following set: $X = \{\langle z_1 \cdots z_n, x \rangle \mid (F_n^A(z_1, \dots, z_n), x) \equiv 1 \pmod{2}\}$. Using the the same approach as the proof of Theorem 6 it is not hard to see that F_n^A can be computed relative X with only a single query. \square

Using the same idea we can prove the following general theorem about oracles for complexity classes other than NP.

Theorem 8 *Let \mathcal{C} be a complexity class and the set $A \leq_m^p$ -complete for \mathcal{C} .*

1. *If \mathcal{C} is closed under \leq_T^p -reductions then $\text{FP}^{\mathcal{C}} = \text{FP}^A \subseteq \text{FEQP}^{A[1]} = \text{FEQP}^{\mathcal{C}[1]}$.*
2. *If \mathcal{C} is closed under \leq_{tt}^p -reductions then $\text{FP}_{\parallel}^{\mathcal{C}} = \text{FP}_{\parallel}^A \subseteq \text{FEQP}^{A[1]} = \text{FEQP}^{\mathcal{C}[1]}$.*

Proof: Let f be the function we want to compute relative to A . Without loss of generality we assume that $|f(z)| = |z|^c$ for some c depending on f . As before we construct the following set:

$$X = \{\langle z, y \rangle \mid (f(z), y) \equiv 1 \pmod{2}, \text{ and } |y| = |z|^c\}.$$

As in Theorem 7 it follows that $f(z)$ is computable with one quantum query to X . Since \mathcal{C} is closed under \leq_T^p -reductions and $X \leq_T^p A$, it follows that $X \in \mathcal{C}$. Furthermore, since A is \leq_m^p -complete for \mathcal{C} it also follows that $X \leq_m^p A$. Thus the quantum query can be made to A itself instead of X . The proof of the second part of the theorem is analogous to the first. \square

This last theorem gives us immediately the following two corollaries about quantum computation with oracles for some known complexity classes.

Corollary 2

$$\begin{aligned} \text{FP}^{\text{PSPACE}} &\subseteq \text{FEQP}^{\text{PSPACE}[1]} \\ \text{FP}^{\text{EXP}} &\subseteq \text{FEQP}^{\text{EXP}[1]} \\ \text{FP}^{\Delta_i^p} &\subseteq \text{FEQP}^{\Delta_i^p[1]} \end{aligned}$$

for the Delta levels Δ_i^p in the polynomial time hierarchy.

Corollary 3

$$\begin{aligned} \text{FP}_{\parallel}^{\text{PP}} &\subseteq \text{FEQP}_{\parallel}^{\text{PP}[1]} \\ \text{FP}_{\parallel}^{\Theta_i^p} &\subseteq \text{FEQP}_{\parallel}^{\Theta_i^p[1]} \end{aligned}$$

with $\Theta_{i+1}^p = \text{P}_{\parallel}^{\Sigma_i^p}$.

The first corollary holds in particular for $A = \text{QBF}$ (the set of true quantified Boolean formulae) which is PSPACE-complete. Observe also that the situation is quite different in the classical setting, since for EXP-complete sets the above is simply not true.

5 Conclusions and Open Problems

We have combined techniques from complexity theory with some of the known quantum algorithms. In doing so we showed that a quantum computer can compute certain functions with fewer queries than classical deterministic computers. Many questions however remain. Is it possible to get trade-off results between the adaptive class $\text{EQP}^{\text{NP}[k]}$ and the non-adaptive $\text{EQP}_{\parallel}^{\text{NP}[2^k-1]}$ for quantum machines? Are the results we present here optimal? (Especially the recent results on exact searching in an ordered list [26] deserve further analysis as they seem to suggest a reduction of the quantum query complexity of Theorems 4 and 6 by a factor of two.)

What can one deduce from the assumption that $\text{P}^{\text{NP}} \subseteq \text{EQP}^{\text{NP}[1]}$? Is it true that for any set A we have $\text{P}^A \subseteq \text{EQP}^{A[1]}$ or are there sets where this is not true? A random set would be a good candidate where more than 1 quantum query is necessary.

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References

- [1] M. Agrawal and V. Arvind. Quasi-linear truth-table reductions to p -selective sets. *Theoretical Computer Science*, 158:361–370, 1996.
- [2] A. Ambainis. A better lower bound for quantum algorithms searching an ordered list. quant-ph report 9902053, Los Alamos archive, 1999.
- [3] A. Amir, R. Beigel, and W. I. Gasarch. Some connections between bounded query classes and nonuniform complexity. In *Proceedings of the 5th Annual Conference on Structure in Complexity Theory*, pages 232–243, 1990.
- [4] A. Amir and W. I. Gasarch. Polynomial terse sets. *Information and Computation*, 77(1):37–56, Apr. 1988.
- [5] J. L. Balcázar, J. Díaz, and J. Gabarró. *Structural Complexity*, volume I. Springer-Verlag, 1988.
- [6] J. L. Balcázar, J. Díaz, and J. Gabarró. *Structural Complexity*, volume II. Springer-Verlag, 1990.
- [7] R. Beigel. *Query-limited reducibilities*. PhD thesis, Stanford University, 1987.
- [8] R. Beigel. NP-hard sets are P-superterse unless $\text{R} = \text{NP}$. Technical Report 88-4, Johns Hopkins University, 1988.
- [9] R. Beigel. Bounded queries to SAT and the Boolean hierarchy. *Theoretical Computer Science*, 2(84):199–223, 1991.
- [10] R. Beigel, R. Chang, and M. Ogiwara. A relationship between difference hierarchies and relativized polynomial hierarchies. *Mathematical Systems Theory*, 26(3):293–310, 1993.
- [11] R. Beigel and W. I. Gasarch. On the complexity of finding the chromatic number of a recursive graph I: The bounded case. *Annals of Pure and Applied Logic*, 45(1):1–38, Nov. 1989.
- [12] R. Beigel, W. I. Gasarch, J. Gill, and J. C. Owings, Jr. Terse, superterse and verbose sets. *Information and Computation*, 103:68–85, 1993.
- [13] R. Beigel, M. Kummer, and F. Stephan. Approximable sets. *Information and Computation*, 120(2):304–314, 1995.
- [14] C. H. Bennett, E. Bernstein, G. Brassard, and U. Vazirani. Strengths and weaknesses of quantum computing. *SIAM Journal on Computing*, 26(5):1510–1523, Oct. 1997. Also on the quant-ph archive, report no. 9701001.
- [15] E. Bernstein and U. Vazirani. Quantum complexity theory. *SIAM Journal on Computing*, 26(5):1411–1473, 1997.
- [16] A. Berthiaume. Quantum computation. In A. L. Selman and L. A. Hemaspaandra, editors, *Complexity Theory Retrospective, In Honor of Juris Hartmanis on the Occasion of His Sixtieth Birthday, July 5, 1988*, volume 2, pages 23–51. Springer-Verlag, Heidelberg, 1997. Also available via Berthiaume’s home page at <http://andre.cs.depaul.edu/Andre/>

- [17] H. Buhrman and L. Fortnow. Two queries. In *Proceedings of the 13th IEEE Conference on Computational Complexity*, pages 13–19, 1998.
- [18] H. Buhrman and R. de Wolf. Lower bounds for quantum search and derandomization. quant-ph report 9811046, Los Alamos archive, 1998.
- [19] S. R. Buss and L. Hay. On truth-table reducibility to SAT. *Information and Computation*, 91(1):86–102, Mar. 1991.
- [20] R. Cleve, A. Ekert, C. Macchiavello, and M. Mosca. Quantum algorithms revisited. *Proceedings of the Royal Society of London A*, 454:339–354, 1998. Also on the quant-ph archive, report no. 9708016.
- [21] S. A. Cook. The complexity of theorem-proving procedures. In *Proceedings of the 3rd ACM Symposium Theory of Computing*, pages 151–158, Shaker Heights, Ohio, 1971.
- [22] W. van Dam. Two classical queries versus one quantum query. quant-ph report 9806090, Los Alamos archive, 1998.
- [23] D. Deutsch. Quantum theory, the Church-Turing principle and the universal quantum computer. *Proceedings of the Royal Society of London A*, 400:97–117, 1985.
- [24] D. Deutsch and R. Jozsa. Rapid solutions of problems by quantum computation. *Proceedings of the Royal Society of London A*, 439:553–558, 1992.
- [25] E. Farhi, J. Goldstone, S. Gutmann, and M. Sipser. A limit on the speedup of quantum computation for insertion into an ordered list. quant-ph report 9812057, Los Alamos archive, 1998.
- [26] E. Farhi, J. Goldstone, S. Gutmann, and M. Sipser. Invariant quantum algorithms for insertion into an ordered list. quant-ph report 9901059, Los Alamos archive, 1999.
- [27] L. Fortnow and J. Rogers. Complexity limitations on quantum computation. In *Proceedings of the 13th IEEE Conference on Computational Complexity*, pages 202–209, 1998. Also on the Los Alamos cc.CC archive, report no. 9811023.
- [28] M. R. Garey and D. S. Johnson. *Computers and Intractability: A Guide to the Theory of NP-Completeness*. W.H. Freeman, San Francisco, 1979.
- [29] E. Hemaspaandra, L. A. Hemaspaandra, and H. Hempel. A downward translation in the polynomial hierarchy. In *14th Annual Symposium on Theoretical Aspects of Computer Science*, volume 1200 of *Lecture Notes in Computer Science*, pages 319–328, Lübeck, Germany, 1997. Springer.
- [30] D. S. Johnson. A catalogue of complexity classes. In J. van Leeuwen, editor, *Handbook of Theoretical Computer Science*, volume A, pages 67–161. Elsevier, Amsterdam, 1990.
- [31] J. Kadin. The polynomial time hierarchy collapses if the Boolean hierarchy collapses. *SIAM Journal on Computing*, 17(6):1263–1282, 1988.
- [32] L. A. Levin. Universal sorting problems. *Problemy Peredaci Informacii*, 9:115–116, 1973. In Russian. English translation in *Problems of Information Transmission* **9**, pages 265–266.
- [33] M. Ogihara. Polynomial-time membership comparable sets. *SIAM Journal on Computing*, 24(5):1068–1081, 1995.
- [34] A. L. Selman. A taxonomy of complexity classes of functions. *Journal of Computer and System Sciences*, 48(2):357–381, Apr. 1994.
- [35] S. Toda. On polynomial-time truth-table reducibility of intractable sets to p -selective sets. *Mathematical Systems Theory*, 24:68–82, 1991.
- [36] K. W. Wagner. Bounded query classes. *SIAM Journal on Computing*, 19(5):833–846, 1990.