The Expected Size of Heilbronn's Triangles

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Abstract

Heilbronn's triangle problem asks for the least Δ such that n points lying in the unit disc necessarily contain a triangle of area at most Δ. Heilbronn initially conjectured $\Delta = O(1/n^2)$. As a result of concerted mathematical effort it is currently known that there are positive constants c and C such that $c \log n/n^2 \leq \Delta \leq C/n^{8/7-\epsilon}$ for every constant $\epsilon > 0$. We resolve Heilbronn's problem in the expected case: If we uniformly at random put n points in the unit disc then (i) the area of the smallest triangle has expectation $\Theta(1/n^3)$; and (ii) the smallest triangle has area $\Theta(1/n^3)$ with probability almost one. Our proof uses the incompressibility method based on Kolmogorov complexity.

1 Introduction

An old problem by Heilbronn is as follows: Let $x_1, \ldots, x_n$ be n points in the unit disc in the plane. Denote by $A_k(x_1, \ldots, x_n)$ the smallest area of all the polygons induced by the point sets $\{x_{i_1}, \ldots, x_{i_k}\}$. Put

$$g_k(n) = \max_{x_1, \ldots, x_n} A_k(x_1, \ldots, x_n).$$

H.A. Heilbronn (1908–1975) asked for the exact value or approximation of $\Delta := g_3(n)$: the maximum possible area of the smallest induced triangle. The list [1, 2, 3, 5, 8, 9, 10, 11, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25] is a selection of papers dealing with the problem. Obviously, the value of Δ will change only by small constant factors considering circles or squares of unit area, and it has become customary to consider the unit square. A brief history is as follows. Heilbronn observed the trivial upper bound $\Delta = O(1/n)$ and conjectured that $\Delta = O(1/n^2)$, and P. Erdős proved that this conjecture—if true—would be tight since $\Delta = \Omega(1/n^2)$ [16]. The first nontrivial result due to K.F. Roth in 1951 established the upper bound $\Delta = O(1/(n\sqrt{\log \log n}))$ [16], which was improved in 1972 by W.M. Schmidt to $O(1/(n\sqrt{\log n}))$ [21] and in the same year by Roth first to $O(1/n^{1.105-\epsilon})$ [17] and then to $\Delta = O(1/n^{1.117-\epsilon})$ for every $\epsilon > 0$ by [18]. Roth simplified his arguments in 1973 and 1976 [19, 20]. Exact values of Δ for $n \leq 15$ were studied in [8, 23, 24, 25]. In 1981, J. Komlós, J. Pintz, and E. Szemerédi [10] improved Roth's upper bound to $O(1/n^{8/7-\epsilon})$, using the simplified arguments of Roth. The really surprising news came in 1982 when the same authors [11] proved a lower bound $\Omega(\log n/n^2)$, refuting Heilbronn's original conjecture. Literally speaking this disproves Heilbronn's conjecture; but "that would be a little harsh and Szemerédi believes that [the lower bound in] [11] is perhaps the best possible" [6, 7]. In 1997 C. Bertram-Kretzberg, T. Hofmeister, and H. Lefmann [3] gave an algorithm that finds a $\log n/n^2$ area triangle for fixed n using a discretization of the problem. Recently G. Barequet [1] studied d-dimensional versions of Heilbronn's problem ($d > 2$).

This Paper: Heilbronn's conjecture is amply correct in the expected case, if the points are thrown in the unit square uniformly at random.

THEOREM 1 *For a uniformly random distribution of n points in the unit square the smallest triangle has expected area of size $\Theta(1/n^3)$. The smallest triangle has area $\Theta(1/n^3)$ with probability almost one.*

This follows directly from Corollaries 2 and 4 of Theorems 2 and 3 to be shown later, respectively. Our results can be used to derive related ones for polygons and multidimensional versions of Heilbronn's problem which will be the subject of a forthcoming paper. The webpage http://www.mathsoft.com/asolve/constant/hlb/hlb.html

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is devoted exclusively to Heilbronn's triangle problem, and Hans Arnold Heilbronn's biographical webpage is http://www-groups.dcs.st-and.ac.uk/~history/Mathematicians/Heilbronn.html.

2 The Incompressibility Method

The incompressibility of individual random objects yields the simple but powerful proof technique used in this paper: *the incompressibility method.* This method is a general purpose tool that can be used to prove lower bounds on computational problems, to obtain combinatorial properties of concrete objects, and to analyze the average complexity of an algorithm. Since the early 1980's, the incompressibility method has been successfully used to solve many well-known questions that had been open for a long time and to supply new simplified proofs for known results. For a survey of the method, see chapter 6 of [14], and for some recent developments, see [12, 4].

Kolmogorov complexity: We define incompressibility in terms of Kolmogorov's robust notion of descriptional complexity [13]. Informally, the Kolmogorov complexity $C(x)$ of a binary string x is the length of the shortest binary program (for a fixed reference universal machine) that prints x as its only output and then halts. A string x is *incompressible* if $C(x)$ is at least $|x|$, the approximate length of a program that simply includes all of x literally. Similarly, a string x is "*nearly*" incompressible if $C(x)$ is "almost as large as" $|x|$.

The appropriate standard for "almost as large" above can depend on the context, a typical choice being "$C(x) \geq |x| - O(\log |x|)$".

Similarly, the *conditional* Kolmogorov complexity of x with respect to y, denoted by $C(x|y)$, is the length of the shortest program that, *with extra information* y, prints x. And a string x is incompressible or nearly incompressible *relative to* y if $C(x|y)$ is large in the appropriate sense.

There are a few well-known facts about these notions that we will use freely, sometimes only implicitly. Proofs and elaboration, when they are not sufficiently obvious, can be found in the literature especially [14]. The simplest is that, both absolutely and relative to any fixed string y, there are incompressible strings of every length, and that *most* strings are nearly incompressible, by *any* standard. [1] Another easy one is that significantly long subwords of an incompressible string are themselves nearly incompressible by *any* standard, even relative to the rest of the string. [2] We need the following lemma. The proof is by simple counting.

LEMMA 1 *Let c be a positive integer. For every fixed y, every finite set A of contains at least* $(1-2^{-c})|A|+1$ *elements x with* $C(x|A,y) \geq \lfloor \log |A| \rfloor - c$.

If we are given A then we can simply enumerate its elements (in say lexicographical order) using an $O(1)$-bit program. Hence the complexity $C(x|A,y) \leq \log |A| + O(1)$.

Asymptotic formula: In the remainder of the paper we will need the asymptotic expression

$$\log \binom{K}{n} \to n \log \frac{K}{n} + n \log e - \frac{1}{2} \log n + O(1) \quad (1)$$

for n fixed and $K \to \infty$. Namely, $\log \binom{K}{n} = n \log(K/n) + R$ where $R = (K-n)\log(K/(K-n)) + \frac{1}{2}\log(K/(n(K-n))) + O(1)$, see [14], p. 10. Using $(1-n/K)^K \to e^{-n}$ for $K \to \infty$ in the first term of R we see that $R \to n \log e - \frac{1}{2} \log n + O(1)$ for n fixed and $K \to \infty$.

Incompressibility method: In a typical proof using the incompressibility method, one first chooses an incompressible (individually random) object from the class under discussion or an object with small enough randomness deficiency. [3] This object is effectively incompressible. The argument invariably says that if a desired property does not hold, then the object can be compressed. This yields the required contradiction. In addition, since the overwhelming majority of objects have small randomness deficiency, the desired property usually also holds on average.

3 Grid and Pebbles

For our analysis of Heilbronn's problem we first consider a discrete version based on a $K \times K$ grid and obtain the general result for the continuous situation by

[1] By a simple counting argument one can show that whereas some strings can be enormously compressed, like strings of the form $11\ldots1$, the majority of strings can hardly be compressed at all. For every n there are 2^n binary strings of length n, but only $\sum_{i=0}^{n-1} 2^i = 2^n - 1$ possible shorter descriptions. Therefore, there is at least one binary string x of length n such that $C(x) \geq n$. Similarly, for every length n and any binary string y, there is a binary string x of length n such that $C(x|y) \geq n$.

[2] Strings that are incompressible are patternless, since a pattern could be used to reduce the description length. Intuitively, we think of such patternless sequences as being random, and we use "random sequence" synonymously with "incompressible sequence." It is possible to give a rigorous formalization of the intuitive notion of a random sequence as a sequence that passes all effective tests for randomness, see for example [14].

[3] Randomness deficiency measures how far the object falls short of the maximum possible Kolmogorov complexity.

taking the limit for $K \to \infty$. So, consider a $K \times K$ grid on the unit square. Call the resulting axis-parallel $2K$ lines *grid lines*. Their crossing points are called *grid points*. We place n points on grid points. These n points will be referred to as *pebbles* to avoid confusion with grid points or other geometric points arising in the discussion.

There are $\binom{K^2}{n}$ ways to put n *unlabeled* pebbles on the grid where at most one pebble is put on each grid point. [4] By Lemma 1, for every function $\delta(\cdot)$ every arrangement $x_1, \ldots, x_n$ out of at least a fraction of $1 - 1/2^{\delta(n)}$ of all arrangements of n pebbles on the grid satisfies:

$$C(x_1, ..., x_n | n, K) \geq \log \binom{K^2}{n} - \delta(n). \qquad (2)$$

We first look at a simple argument giving a weak but already nontrivial result. A proof by Erdős reported in [16] shows that for the special case of $p \times p$ grids, where p is a prime number, there are necessarily arrangements of p pebbles with every pebble placed on a grid point such that no three pebbles are collinear. Therefore, the smallest triangle in such an arrangement has area at least $1/2p^2$. This implies that the triangle constant Δ is not $o(1/n^2)$. Using the incompressibility method, it is easy (and used later on in the paper) to obtain related but weaker results which, however, hold for *almost all* arrangements of $n \ll K$ pebbles on a $K \times K$ grid for all large enough K.

Lemma 2 *Let n and K be large enough and let K exceed n enough to validate the argument below. Then, for every arrangement of n pebbles on a $K \times K$ grid so that inequality (2) holds for some $\delta(n) = O(n)$, no three pebbles u, v, w out of $x_1, \ldots, x_n$ can be on a straight line (in the real Euclidean plane).*

Proof. Suppose that pebbles u, v and w are on a straight line in the Euclidean plane spanned by coordinates x, y. Then we can specify u, v and w by first giving u, v, from which we can compute the coefficients a, b, c of the equation $ax + by = c$ of the straight line that is incident on the three pebbles. Given this line, to specify w we only need to give the number of grid points incident on this line in between u (or v) and w. Since all of the three pebbles are placed on grid points of the $K \times K$ grid, this number is less than K. Therefore, given K we can describe the places of the three pebbles in $5 \log K$ bits. Consequently, to describe $x_1, \ldots, x_n$ given n and K we only need:

- A description of this discussion in $O(1)$ bits. Since we know the values of n and K the following parts of the encoding can be decoded uniquely:
- A description of the locations of u, v, w on the grid in $5 \log K$ bits; and
- The arrangement of the other $n - 3$ pebbles in $\log \binom{K^2}{n-3}$ bits.

Altogether this description takes

$$A := \log \binom{K^2}{n-3} + 5 \log K + O(1)$$

bits which by definition cannot be less than the complexity assumed by (2):

$$A \geq \log \binom{K^2}{n} - \delta(n).$$

Using (1) we obtain:

$$3 \log n \geq \log K - \delta(n) + O(1).$$

For every $\delta(n)$ we obtain a contradiction for large enough $K \gg n$. $\square$

With $\delta(n) = \log \log n$ we obtain the contradiction in the proof already for

$$n \log^{1/3} n = o(K^{1/3}). \qquad (3)$$

Moreover, the smallest triangle induced by three non-collinear pebbles on the grid has area at least $1/(2K^2)$. That is, consider an arrangement of n pebbles on a $K \times K$ grid satisfying (2) with $\delta(n) = \log \log n$, and (3). Then, the smallest triangle formed by three pebbles has area at least $\Omega(1/(n^6 \log^{2+\epsilon} n))$ where $\epsilon > 0$ is an arbitrarily small constant.

Lemma 3 *Let n and K be large enough and let K exceed n enough to validate the argument below. Then, for every arrangement of n pebbles on a $K \times K$ grid so that inequality (2) holds for some $\delta(n) = O(n)$, no two pebbles u, v out of $x_1, \ldots, x_n$ can be on the same grid line.*

Proof. Suppose two pebbles are placed on the same (say horizontal) grid line. To describe $x_1, \ldots, x_n$ given n and K we only need:

[4] We count only distinguishable distributions without regard for the identities of the placed pebbles. With every arrangement having the same probability $1/\binom{K^2}{n}$, the result is a probability distribution known in physics as the *Fermi-Dirac* distribution. Clearly, the restriction that no two pebbles can be placed on the same grid point is no restriction anymore when we let K grow unboundedly.

- A description of this discussion in $O(1)$ bits. Since we know the values of n and K the following parts of the encoding can be decoded uniquely:
- A description of the grid line containing u, v in $\log K$ bits;
- A description of the locations of u, v on the grid in $2 \log K$ bits; and
- The arrangement of the other $n-2$ pebbles in $\log \binom{K^2}{n-2}$ bits.

Altogether this description takes $A := \log \binom{K^2}{n-2} + 3 \log K + O(1)$ bits which by definition cannot be less than the complexity assumed by (2), that is,

$$A \geq \log \binom{K^2}{n} - \delta(n).$$

Using (1) we obtain:

$$2 \log n \geq \log K - \delta(n) + O(1).$$

For every $\delta(n)$ we obtain a contradiction for large enough $K \gg n$. □

4 Lower Bound

Assume the grid/pebble terminology. We extend the approach above to show that if there is a too small triangle then we can compress the description of the arrangement to below the complexity stated in (2).

THEOREM 2 *Let n and K be large enough and let K exceed n enough to validate the argument below. Then, for every arrangement of n pebbles on a $K \times K$ grid so that inequality (2) holds for some $\delta(n) = O(n)$, the smallest triangle formed by three points has area $\Omega(1/(2^{\delta(n)} n^3))$.*

PROOF. Assume grid-coordinates $\{1, \ldots, K\} \times \{1, \ldots, K\}$ such that two adjacent grid points in a row or column are $1/K$ apart. Let the smallest triangle formed by three out of the n pebbles on the $K \times K$ grid have area Δ. Suppose that pebbles $u = (u_1, u_2)$, $v = (u_1 + v_1, u_2 + v_2)$ and $w = (u_1 + w_1, u_2 + w_2)$ form the smallest triangle. Without loss of generality we assume that the longest side connects the pair of pebbles u and v. This side, denoted by (u, v), corresponds to the vector $(v - u) := (v_1, v_2)$ and its real Euclidean length is $l := \sqrt{v_1^2 + v_2^2}/K$. Then, w lies on one of the two line segments that lie at real Euclidean distance $h := 2\Delta / l$ parallel to the side (u, v), as in Figure 1. The number of grid points on each of these three line segments (including one endpoint), that each pass through some grid point, and have identical length and orientation, is given by $g = \gcd(v_1, v_2)$.

The inner-product of the vector $(v - u)^{\perp} := (v_2, -v_1)$ (orthogonal to the vector $(v - u)$) with the vector $(w - u) := (w_1, w_2)$ equals the product of their lengths times the cosine of the enclosed angle. The real Euclidean length of the vector $(v_2, -v_1)$ is l while the real Euclidean length of vector (w_1, w_2) times the cosine of the enclosed angle equals h. Therefore, the inner product is two times the area of the triangle (u, v, w) and we have $2K^2\Delta = |v_2 w_1 - v_1 w_2|$, which is necessarily a non-zero multiple of g. [5] Given the long side (u, v) we can determine the number g of grid points incident on (u, v). Since $2K^2\Delta$ is a multiple of g, we know that $2K^2\Delta = fg$ where f is integer.

Thus, we can describe w as follows: we use at most $\log 2g$ bits to encode the grid point of w as one of the the $2g$ grid points on the two parallel line segments at distance h of (u, v). If necessary we pad this description to length $\log 2g$ in order to be able to parse this segment using the known value of g. Subsequently we encode $f = 2K^2\Delta/g$ in $\log f$ bits. Altogether this takes $\log 2g + \log f = 1 + \log(2K^2\Delta)$ bits.

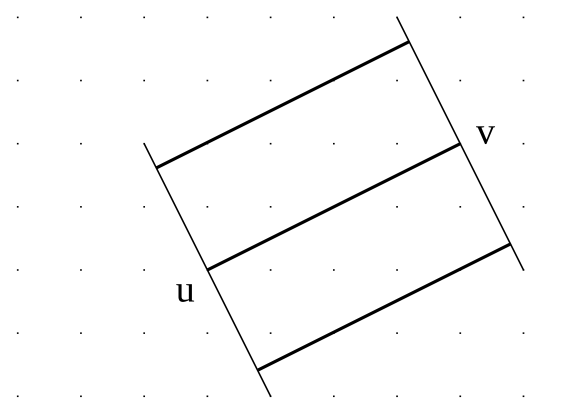

Figure 1: Smallest triangle based on pebbles u, v.

Therefore, given n and K we can specify $x_1, \ldots, x_n$ by listing:

- A description of this discussion in $O(1)$ bits, and since we know the values of n and K the following parts of the encoding can be decoded uniquely:

[5] The quantity $2K^2\Delta$ equals the number of grid points contained in one of the two rectangles in Figure 1, possibly up to an *additional term* of order of circumference of the rectangle. Intuitively, we can describe w by giving its index in an enumeration of the set of these grid points. But we have to deal with the grid geometry. Therefore, we give w's index in the g gridpoints on one of the line segments at distance h parallel to (u, v) in Figure 1, together with a precise expression of h in terms of grid coordinates.

- The concatenated descriptions of the ordered list of coordinates of $x_1, \dots, x_n$ with those of w deleted in $\log \binom{K^2}{n-1}$ bits;
- The indices of u, v in the ordered list $\{x_1, \dots, x_n\}$ $-\{w\}$ in $2 \log n$ bits;
- The grid point containing w which is one of the $2g$ grid points on the two line segments at distance h as follows: We first give the index of the grid point containing w in enumeration order in $\log 2g$ bits. This code segment can be uniquely determined since from the previous description items we can reconstruct the line segment (u, v) and determine g as the number of grid points on (u, v). Finally, we give $f = 2K^2\Delta/g$ in $\log f$ bits. Since this is the last part of overall description it doesn't need to be self-delimiting. In total this description item takes at most $\log(\Delta K^2) + O(1)$ bits.

The first description item is coded self-delimiting and contains the program to do the following reconstruction: Determine the second description item by computing its length from n and K that are provided for free. Reconstruct the grid coordinates of $x_1, \dots, x_n$ with w deleted from the second description item. Reconstruct the grid point containing w as follows: Use n to determine the length of the second description item and its parts and reconstruct the indices of u and v in the list $x_1, \dots, x_n$ with w deleted. Determine u and v which gives the long side (u, v) of the triangle. Use (u, v) to determine g. Use g to determine the first $\log 2g$ bits of the fourth description item. Reconstruct the index j $(1 \leq j \leq 2g)$ of the grid point that contains w on the two possible line segments. Use the final part of the fourth description item (the last part of the overall description) to reconstruct f. Compute $2\Delta = fg/K^2$ using K as given. Subsequently, compute the length l of (u, v) and compute $h = 2\Delta/l$ *precisely* in terms of the unit grid distance $1/K$. Substitution of the above values of Δ and l expressed in grid coordinates yields:

$$h = \frac{|v_2 w_1 - v_1 w_2|}{K\sqrt{v_1^2 + v_2^2}}.$$

Now use (u, v) and h to determine the two parallel line segments on which w is located. Use the previously computed index j to determine the grid point containing w. Altogether the description takes at most:

$$\log \binom{K^2}{n-1} + 2\log n + \log(\Delta K^2) + O(1)$$

bits, which must be at least $\log \binom{K^2}{n} - \delta(n)$ by inequality (2). By approximation (1),

$$\log \binom{K^2}{n} - \log \binom{K^2}{n-1} \to \log \frac{K^2}{n} + O(1)$$

for large enough fixed n and $K \to \infty$. Therefore, $\log \Delta \geq -3 \log n - \delta(n) + O(1)$. Since the argument holds for every K, letting $K \to \infty$ proves the theorem. □

By Lemma 1 the probability concentrated on the set of arrangements satisfying (2) is at least $1 - 1/2^{\delta(n)}$.

Corollary 1 *Putting n points in the unit square uniformly at random, the smallest triangle formed by three points has area $\Omega(1/(2^{\delta(n)} n^3))$ with probability at least $1 - 1/2^{\delta(n)}$.*

In the particular case of $\delta(n) \leq 1$ the probability concentrated on arrangements satisfying (2) is at least $\frac{1}{2}$.

Corollary 2 *Putting n points in the unit square uniformly at random, the smallest triangle formed by three points has expected area $\Omega(1/n^3)$. The smallest triangle has area $\Omega(n^3)$ with probability almost one.*

5 Upper Bound

Every two pebbles out of an incompressible arrangement of n pebbles on a $K \times K$ grid are incident with a separate line by Lemma 2. There are precisely $\binom{n}{2}$ such lines altogether. The two pebbles defining a line form a triangle with every third pebble. As before write Δ for the area of the smallest triangle. Every side of every triangle formed by three pebbles has length at most $\sqrt{2}$ (the length of a diagonal of the unit square). Every line incident on two pebbles defines a strip of width at least $\Delta/\sqrt{2}$ on each side where no pebbles can be placed. Namely, a pebble in the strip together with the two original pebbles would form a triangle of area less than Δ. Our goal is to show that already $n/2$ of the pebbles induce $\Omega(n^2)$ lines which are in some sense far enough apart so that these forbidden strips don't overlap. Then the number of grid points that are contained in these forbidden strips is so large that the number of grid points on which we can place the remaining $n/2$ pebbles gets severely restricted. In fact, if Δ rises then the potential places to put the pebbles get restricted so much that the complexity of the arrangement can be compressed to below the assumed incompressibility, which yields the aimed-for contradiction. This argument is so precise that it turns out that for small randomness deficiency

$\delta(n)$ the upper bound is the same order of magnitude as the previously proven lower bound.

THEOREM 3 *Let n and K be large enough and let K exceed n enough to validate the argument below. Then, for every arrangement of n pebbles on a $K \times K$ grid so that inequality (2) holds for some $\delta(n) < (2-\epsilon)\log n$ for any positive constant ϵ. Then, the smallest triangle formed by three points has area $O(\delta(n)/n^3)$.*

PROOF. Divide the unit square by a horizontal grid line into an upper and a lower half each of which contains $n/2 \pm 2$ pebbles. This is possible since we have shown that there are no three collinear pebbles in Lemma 2. For convenience we assign the possible (two) pebbles on the dividing line to the upper and lower halves so that each half has exactly $n/2$ pebbles. We write *upper line* for a geometric line incident with two pebbles in the upper half.

CLAIM 1 Every horizontal grid line in the lower half intersects $\Omega(n^2)$ upper lines.

PROOF. There are $\binom{n/2}{2}$ upper lines. Take the top half to be the larger half so that it has area at least 1/2. Divide the top half into five columns of equal width of 1/5 and five rows of equal width of at least 1/5 each. Standard geometric arguments show that all horizontal grid lines in the bottom half are crossed by every upper line determined by a pebble in the upper rectangle and a pebble in the lower rectangle of the middle column. To prove the claim we only need to show that the top and bottom rectangles of the middle column contain $\Omega(n)$ pebbles each. To the contrary, suppose that some rectangle contains $m := o(n)$ pebbles. Since the area of the rectangle is at least 1/50 it contains at least $K^2/50$ grid points. [6] Therefore we can describe the arrangement of n pebbles by separately describing the pebbles in the sparsely populated rectangle and the pebbles in the remainder of the unit square. This takes at most

$$\log \binom{49K^2/50}{n-m}\binom{K^2/50}{m}$$

bits which goes to [7]

$$(n-m)\log\frac{49K^2/50}{n-m} + m\log\frac{K^2/50}{m} + n\log e - \frac{1}{2}\log n,$$

for $K \to \infty$ with n fixed, by (1). This number must be at least the complexity of the arrangement up to an additive constant term. Therefore, by (2)

$$(n-m)\log\frac{49K^2/50}{n-m} + m\log\frac{K^2/50}{m} \geq n\log\frac{K^2}{n} - \delta(n),$$

which implies

$$\delta(n) \geq (n-m)\log\frac{50(n-m)}{49} + m\log 50m.$$

But our contradictory assumption put $m = o(n)$ which implies $\delta(n) = \Omega(n \log n)$ which contradicts $\delta(n) = O(\log n)$ in the statement of the theorem. □

The area of the smallest triangle is $\Delta = \Omega(1/n^3)$ by Theorem 2. Every upper line is based on two pebbles which together with any third pebble forms a triangle of area at least Δ. Because of the orientation of every upper line with which it intersects a lower horizontal grid line this eliminates $\Omega(K\Delta)$ adjacent potential pebble placement grid point positions on the horizontal grid line on both sides of the intersection. The maximal number of grid points are eliminated (up to order of magnitude) if every horizontal lower grid line that contains a pebble has $\Omega(n^2)$ intersections with upper lines that are pairwise so far apart that the sets of forbidden grid points associated with intersections pairwise don't overlap.

CLAIM 2 The outermost elements of any six intersections of the upper lines with a horizontal grid line passing through some pebble in the bottom half have distance at least $d = \Omega(1/n^{3-\epsilon/5})$ with $\epsilon > 0$ constant.

PROOF. We denote an upper line by (p, p') where p and p' are the pebbles in the upper half that define the line. Let d be the least distance between the outermost elements of any six intersections of the upper lines with a horizontal grid line passing through some pebble in the bottom half. [8] Fix the position of a pebble in the bottom half, say p_0, and thereby the incident horizontal grid line. Suppose that lines $(p_1, p_2), (p_3, p_4), (p_5, p_6), (p_7, p_8), (p_9, p_{10})$, and (p_{11}, p_{12}) intersect the horizontal grid line incident with pebble p_0, within distance d between the outermost intersections. That is, all six intersections are located on a line segment of length d of the horizontal grid line determined by pebble p_0. Given n and K we can reconstruct the coordinates of pebbles p_4, p_6, p_8, p_{10}, and p_{12} from the arrangement of the other $n-5$ pebbles on the grid by listing the following description:

[6] Plus or minus $O(K)$ grid points which we ignore in the remainder of the argument.

[7] We ignore constant terms in the remainder of this proof.

[8] The following proof can be done for a variable number of intersections. But precisely *six* intersections turn out to be necessary and sufficient.

- A description of this discussion in $O(1)$ bits, and since we know the values of n and K the following parts of the encoding can be decoded uniquely:
- The indices of pebbles $p_1, p_2, p_3, p_5, p_7, p_9, p_{11}$, and p_0 in the list $x_1, \ldots, x_n$, in $8 \log n$ bits;
- The locations of the five points p_4, p_6, p_8, p_{10}, and p_{12} on the five lines (p_3, p_4), (p_5, p_6), (p_7, p_8), (p_9, p_{10}), and (p_{11}, p_{12}), relative to their intersections with the horizontal grid line, in $5 \log K$ bits (they are the unique grid points in circles with radius $1/(4K)$ centered on the computed geometric points); and
- The geometric coordinates of the five intersections by the lines (p_3, p_4), (p_5, p_6), (p_7, p_8), (p_9, p_{10}), and (p_{11}, p_{12}) relative to the intersection by the line (p_1, p_2) up to $1/(4K)$ precision in $5 \log dK + O(1)$ bits. Since this is the last part of the description it doesn't need to be self-delimiting. Moreover, we encode the five constituent parts in five blocks of equal size (possibly using padding). We can then simply divide this final description item into five equal blocks.

In total this takes at most

$$A := 8 \log n + 5 \log d + 5 \log K^2$$

bits up to an additional constant term. Together with the description of the remaining $n-5$ pebbles, which we can insert in the description in between the first and the second item (not self-delimiting because we know n and K and can therefore parse this part) the new description must take at least as many bits as the complexity according to inequality (2):

$$\log \binom{K^2}{n-5} + A \geq \log \binom{K^2}{n} - \delta(n)$$

up to some additional constant. Using approximation (1) again:

$$\log \binom{K^2}{n} - \log \binom{K^2}{n-5} \to 5 \log K^2 - 5 \log n$$

up to some additional constant term for large fixed n and $K \to \infty$. Therefore, $5 \log d \geq -13 \log n - \delta(n)$ up to some additional constant, and hence $d = \Omega(2^{(2 \log n - \delta(n))/5}/n^3)$. Substituting $\delta(n) \leq (2 - \epsilon) \log n$ with ϵ a positive constant (by assumption of the theorem) proves the claim. □

Summarizing: by Claim 1, every horizontal grid line in the lower half containing a pebble is intersected by $\Omega(n^2)$ upper lines, and by Claim 2 every interval of length d on every such grid line contains at most six such intersections. Considering only one intersection for every odd interval of length d along a grid line there is at least d distance between consecutive such intersections, and there remain $\Omega(n^2)$ such special intersections per horizontal grid line (that contains a pebble).

As before let Δ be the area of the smallest triangle. Every pebble p'' within distance $2\Delta/\sqrt{2}$ to an upper line (p, p') forms a triangle (p, p', p'') of area less than Δ. Hence, no pebbles in the arrangement except the defining two pebbles p and p' can be within distance $2\Delta/\sqrt{2}$ to an upper line (p, p'). In particular, no grid points within Δ distance to an upper line intersection with a horizontal grid line in the lower half can be used to place a pebble.[9] Of course if the forbidden strips around the upper lines overlap then the total number of forbidden grid points may be low. But if

$$\Delta < \frac{d}{2} \tag{4}$$

then the sets of grid points eliminated by the forbidden strip of the $\Omega(n^2)$ selected upper lines on every horizontal grid line containing a pebble are pairwise disjoint. This eliminates a large number of grid points to potentially place the $n/2$ pebbles in in the lower half. If Δ is large then this restriction allows us to compress the description to below the complexity of (2).

To obtain an upper bound for Δ this way, we will describe the arrangement of the pebbles by encoding their x- and y-coordinates separately. Observe that, to specify the x-coordinate of any one of the $n/2$ pebbles in the lower half, we need only consider $K(1 - \Omega(n^2 \Delta))$ grid points on the appropriate horizontal grid line, provided (4) holds, or only $K(1 - \Omega(n^2 d))$ grid points if (4) doesn't hold. Define B by

$$B := \min\{\Delta, \frac{d}{2}\}. \tag{5}$$

For some constant $c > 0$ (implicit in the Ω-notation) we can describe the x-coordinates of every one of the lower $n/2$ pebbles in

$$\log K + \log(1 - cn^2 B) \tag{6}$$

bits. Therefore, we can describe the n pebbles by giving:

[9] By the construction of the upper lines they intersect the horizontal grid lines in the lower half within an angle of $\pi/8$ of perpendicular. Therefore the forbidden strip covers at least $(2\Delta/\sqrt{2})/\sqrt{2}$ grid points on each side.

- A description of this discussion in $O(1)$ bits, and since we know the values of n and K the following parts of the encoding can be decoded uniquely:

- A description of the set of y coordinates in
$$\log \binom{K}{n}$$
bits to indicate both the n horizontal grid lines containing pebbles (no grid line can contain more than one pebble by Lemma 3).

- A description of the x-coordinates of the pebbles in the upper half using $\log \binom{K}{n/2}$ bits plus $\log(n/2)!$ bits to order them according to the y-coordinates; and

- A description of the x-coordinates in the lower half in $\frac{n}{2} \log K$ bits in ascending order of their y-coordinates.

Altogether this is at most (up to an additional constant term):

$$\log \binom{K}{n} + \frac{n}{2} \log K + \frac{n}{2}(\log K + \log(1 - cn^2 B))$$
$$\to n \log \frac{K^2}{n} + n \log e - \frac{1}{2} \log n + \frac{n}{2} \log(1 - cn^2 B),$$

where the right-hand side follows from (1). This should be at least the complexity as in inequality (2). Thus,

$$\frac{n}{2} \log(1 - cn^2 B) \geq -\delta(n) + O(1).$$

The left-hand side can be rewritten and for $n \to \infty$ has the following asymptotic behavior:

$$\log \left(1 - \frac{cn^3 B/2}{n/2}\right)^{n/2} \to \log e^{-cn^3 B/2}.$$

That is, $cn^3 B \log e \leq 2\delta(n) + O(1)$ so that

$$B \leq \frac{2\delta(n) + O(1)}{cn^3 \log e} = O(\frac{\delta(n)}{n^3}) \quad (= O(\frac{\log n}{n^3})). \quad (7)$$

The last equality follows since $\delta(n) < 2 \log n$ in the statement of the theorem. By (5) this relation must hold with either Δ or $\frac{d}{2}$ substituted for B. Claim 2 tells us that substituting $\frac{d}{2}$ for B doesn't satisfy (7). Hence (7) holds with Δ substituted for B. Since the argument holds for every K, letting $K \to \infty$ proves the theorem. $\square$

Using Lemma 1 again we find:

Corollary 3 *Putting n points in the unit square uniformly at random, the smallest triangle formed by three points has area $O(\delta(n)/n^3)$ with probability at least* $1 - 1/2^{\delta(n)}$.

The expectation of the smallest area of a triangle is thus computed as:

$$\sum_{\delta(n)=0}^{1.9 \log n} \frac{1}{2^{\delta(n)+1}} O\left(\frac{\delta(n)}{n^3}\right) + \frac{1}{n^{1.9}} O\left(\frac{1}{n^{8/7-\epsilon}}\right)$$
$$= O(\frac{1}{n^3})$$

since the area of the smallest triangle is upper bounded by $O(1/n^{8/7-\epsilon})$ for every $\epsilon > 0$ in *all* arrangements [10].

Corollary 4 *Putting n points in the unit square uniformly at random, the smallest triangle formed by three points has expected area $O(1/n^3)$. The smallest triangle has area $O(n^3)$ with probability almost one.*

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