Backdoors to Acyclic SAT*

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Abstract. Backdoor sets contain certain key variables of a CNF formula F that make it easy to solve the formula. More specifically, a *weak backdoor set* of F is a set X of variables such that there exits a truth assignment τ to X that reduces F to a satisfiable formula $F[\tau]$ that belongs to a polynomial-time decidable base class $\mathcal{C}$. A *strong backdoor set* is a set X of variables such that for all assignments τ to X, the reduced formula $F[\tau]$ belongs to $\mathcal{C}$.

We study the problem of finding backdoor sets of size at most k with respect to the base class of CNF formulas with acyclic incidence graphs, taking k as the parameter. We show that

1. the detection of weak backdoor sets is W[2]-hard in general but fixed-parameter tractable for r-CNF formulas, for any fixed $r \geq 3$, and
2. the detection of strong backdoor sets is fixed-parameter approximable.

Result 1 is the the first positive one for a base class that does not have a characterization with obstructions of bounded size. Result 2 is the first positive one for a base class for which strong backdoor sets are more powerful than deletion backdoor sets.

Not only SAT, but also #SAT can be solved in polynomial time for CNF formulas with acyclic incidence graphs. Hence Result 2 establishes a new structural parameter that makes #SAT fixed-parameter tractable and that is incomparable with known parameters such as treewidth and clique-width. We obtain the algorithms by a combination of an algorithmic version of the Erdős-Pósa Theorem, Courcelle's model checking for monadic second order logic, and new combinatorial results on how disjoint cycles can interact with the backdoor set.

1 Introduction

Since the advent of computational complexity in the 1970s it quickly became apparent that a large number of important problems are intractable [16]. This predicament motivated significant efforts to identify tractable special cases. For the propositional satisfiability problem (SAT), dozens of such "islands of tractability" have been identified [14]. Whereas it may seem unlikely that a real-world

* The full version of the paper is available on arXiv [17].

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instance belongs to a known island of tractability, it may be "close" to one. In this paper we study the question of whether we can exploit the proximity of a SAT instance to the island of acyclic formulas algorithmically.

For SAT, the distance to an island of tractability (or *base class*) $\mathcal{C}$ is most naturally measured in terms of the number of variables that need to be instantiated to put the formula into $\mathcal{C}$. Williams *et al.* [32] introduced the term "*backdoor set*" for sets of such variables, and distinguished between weak and strong backdoor sets. A set B of variables is a *weak $\mathcal{C}$-backdoor set* of a CNF formula F if for at least one partial truth assignment $\tau : B \to \{0,1\}$, the restriction $F[\tau]$ is satisfiable and belongs to $\mathcal{C}$. The set B is a *strong $\mathcal{C}$-backdoor set* of F if for every partial truth assignment $\tau : B \to \{0,1\}$ the restriction $F[\tau]$ belongs to $\mathcal{C}$.

1.1 Weak Backdoor Sets

If we are given a weak $\mathcal{C}$-backdoor set of F of size k, we know that F is satisfiable, and we can verify the satisfiability of F by checking whether at least one of the 2^k assignments to the backdoor variables leads to a satisfiable formula that belongs to $\mathcal{C}$. If the base class allows to find an actual satisfying assignment in polynomial time, as is usually the case, we can find a satisfying assignment of F in $2^k n^{O(1)}$ time. Can we find such a backdoor set quickly if it exists? For all reasonable base classes $\mathcal{C}$ it is NP-hard to decide, given a CNF formula F and an integer k, whether F has a strong or weak $\mathcal{C}$-backdoor set of size at most k. On the other hand, the problem is clearly solvable in time $n^{k+O(1)}$. The question is whether we can get k out of the exponent, and find a backdoor set in time $f(k)n^{O(1)}$, i.e., is weak backdoor set detection *fixed-parameter tractable (FPT)* in k? Over the last couple of years, this question has been answered for various base classes $\mathcal{C}$; Table 1 gives an overview of some of the known results. See [18] for a survey.

For general CNF, the detection of weak $\mathcal{C}$-backdoor sets is W[2]-hard for all reasonable base classes $\mathcal{C}$. For some base classes the problem becomes FPT if clause lengths are bounded. All FPT results for weak backdoor set detection in Table 1 are due to the fact that for r-CNF formulas, where $r \geq 3$ is a fixed constant, membership in the considered base class can be characterized by certain obstructions of bounded size. Formally, say that a base class $\mathcal{C}$ has the *small obstruction property* if there is a family $\mathcal{F}$ of CNF formulas, each with a finite number of clauses, such that for every CNF formula F, $F \in \mathcal{C}$ if and only if F contains no subset of clauses isomorphic to a formula in $\mathcal{F}$. Hence, if a base class $\mathcal{C}$ has this property, fixed-parameter tractability for weak $\mathcal{C}$-backdoor set detection for r-CNF formulas can be established by a bounded search tree algorithm.

The base class FOREST is another class for which the detection of weak backdoor sets is W[2]-hard for general CNF formulas (Theorem 3). For r-CNF formulas the above argument does not apply because FOREST does not have the small obstruction property. Nevertheless, we can still show that the weak FOREST backdoor set detection problem is FPT for r-CNF formulas, for every fixed $r \geq 3$ (Theorem 4). This is our first main result.