An Implicit Representation of Chordal Comparabilty Graphs in Linear-Time

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Abstract. Ma and Spinrad have shown that every transitive orientation of a chordal comparability graph is the intersection of four linear orders. That is, chordal comparability graphs are comparability graphs of posets of dimension four. Among other uses, this gives an implicit representation of a chordal comparability graph using $O(n)$ integers so that, given two vertices, it can be determined in $O(1)$ time whether they are adjacent, no matter how dense the graph is. We give a linear-time algorithm for finding the four linear orders, improving on their bound of $O(n^2)$.

1 Introduction

A *partial order* or *poset* relation is a transitive antisymmetric relation. In this paper, we consider the graphical representation of a poset using a directed acyclic and transitive graph. When we say the graph is *transitive*, we mean that whenever $x \rightarrow y \rightarrow z$, $x \rightarrow z$. Whether the partial order is reflexive is irrelevant to our goals, so we only consider loopless graphs. The *comparability relation* of a partial order is the set of pairs that are comparable in the partial order. That is, it is the symmetric closure, where, whenever (a, b) is in the partial order, (b, a) is added to it. The comparability relation has a natural representation as an undirected graph that has an edge ab whenever (a, b) and (b, a) are in the comparability relation; it is obtained by ignoring edge directions in the transitive graph that represents the partial order. And given a comparability graph, it is possible to *transitively orient* it in linear time [MS99], that is, to recover a corresponding partial order.

A *chordal graph* is an undirected graph where each cycle of length four or greater has a *chord*, that is, an edge that is not on the cycle but whose endpoints are both on the cycle.

A co-comparability graph or co-chordal graph is one whose complement is a comparability graph or chordal graph, respectively. Many interesting graph classes are defined by intersecting the comparability, co-comparability, chordal and co-chordal graph classes.

An example is an *interval graph*, which is the intersection graph of a set of intervals on the line, that is, the graph that has one vertex for each of the intervals

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and an edge for each intersecting pair. These are exactly the intersection of the chordal and co-comparability graphs.

A *permutation graph* is defined by a permutation of a linearly ordered set of objects. The vertices are the objects, and the edges are the *non-inversions*, that is, the pairs of objects whose relative order is is the same in the two permutations. These are exactly the intersection of the comparability and co-comparability graphs.

A *split graph* is a graph whose vertices can be partitioned into a clique and an independent set. These are exactly the intersection of the chordal and co-chordal graphs. More information about all graph classes classes mentioned here can be found in [Gol80].

All of these graphs are subclasses of the class of perfect graphs, because comparability graphs and chordal graphs are perfect. Interval graphs can be represented with $O(n)$ integers, numbering the endpoints in left-to-right order and associating each vertex with its endpoint numbers. Adjacency can then be tested in $O(1)$ time by comparing the two pairs of endpoints of the vertices to see if they correspond to intersecting intervals. Similarly, permutation graphs can be represented by numbering the vertices in left-to-right order in two linear orders, and testing adjacency in $O(1)$ time by determining whether the two vertices have the same relative order in both. These are examples of *implicit representations*; for more details see Spinrad's book on the topic of implicit representations of graph classes [Spi03].

A *linear order* is just a special case of a partial order, where the elements are numbered 1 through n, and the relation is the set of ordered pairs $\{(i,j)|i < j\}$. This partial order has $\Theta(n^2)$ elements, but can be represented implicitly by giving the ordering or numbering of the vertices.

It is easy to see that the intersection of two partial orders (the ordered pairs that are common to both) is also a partial order, hence this applies to the intersection of linear orders. In fact, every partial order is the intersection of a set of linear orders [DM41]. A partial order has *dimension* k if there exist k linear orders whose intersection is exactly that partial order. It is easy to see from this that the permutation graphs are just the comparability graphs of two-dimensional partial orders. Two-dimensional partial orders and permutation graphs can be recognized and their representation with two linear orders can be found in linear time [MS99]. In general, k linear orders gives an $O(nk)$ representation, but unfortunately, it is NP-complete to determine whether a partial order has dimension k for $k \geq 3$ [Yan82].

In this paper, we examine *chordal comparability graphs*, that is, the intersection of the class of chordal graphs and the class of comparability graphs. Ma and Spinrad have shown that all chordal comparability graphs are the comparability graphs of partial orders of dimension at most four [MS91, Spi03]. The four linear orders give a way of representing the graph in $O(n)$ space so that for any two vertices, it can be answered in $O(1)$ time whether they are adjacent. Each vertex is labeled with the four position numbers of each vertex in the four linear order, and for two vertices, they are adjacent iff one of them precedes the other